Nonlinear Finite Element Method

08/11/2004

Nonlinear
Finite Element Method

- Lectures include discussion of the nonlinear finite element method.
- It is preferable to have completed "Introduction to Nonlinear Finite Element Analysis" available in summer session.
- If not, students are required to study on their own before participating this course. Reference:Toshiaki.,Kubo. "Introduction: Tensor Analysis For Nonlinear Finite Element Method" (Hisennkei Yugen Yoso no tameno Tensor Kaiseki no Kiso),Maruzen.
- Lecture references are available and downloadable at http://www.sml.k.u-tokyo.ac.jp/members/nabe/lecture2004 They should be posted on the website by the day before scheduled meeting, and each students are expected to come in with a copy of the reference.
- Lecture notes from previous year are available and downloadable, also at http://www.sml.k.u.tokyo.ac.jp/members/nabe/lecture2003 You may find the course title, "Advanced Finite Element Method" but the contents covered are the same I will cover this year.
- I will assign the exercises from this year, and expect the students to hand them in during the following lecture. They are not the requirements and they will not be graded, however it is important to actually practice calculate in deeper understanding the finite element method.
- For any questions, contact me at nabe@sml.k.u-tokyo.ac.jp

Nonlinear Finite Element Method Lecture Schedule

1. 10/ 4 Finite element analysis in boundary value problems and the differential equations
2. 10/18 Finite element analysis in linear elastic body
3. 10/25 Isoparametric solid element (program)
4. 11/ 1 Numerical solution and boundary condition processing for system of linear equations（with exercises）
5. 11/ 8 Basic program structure of the linear finite element method(program)
6. 11/15 Finite element formulation in geometric nonlinear problems(program)
7. 11/22 Static analysis technique、hyperelastic body and elastic-plastic material for nonlinear equations (program)
8. 11/29 Exercises for Lecture7
9. 12/ 6 Dynamic analysis technique and eigenvalue analysis in the nonlinear equations
10. 12/13 Structural element
11. 12/20 Numerical solution— skyline method、iterative method for the system of linear equations
12. 1/17 ALE finite element fluid analysis
13. 1/24 ALE finite element fluid analysis

Boundary Value Problem For Linear Elastic Body

Consider, a boundary value problem[B] for a linear elastic body A found in the figure below. Ω is a region occupied by [B] , and the body A Ω has its boundary $\partial\Omega$. A displacement boundary condition is given on its subset $\partial\Omega D$. When surface force $\boldsymbol{t}$, body force $\rho\boldsymbol{g}$ are acted on such systems, find the displacement $\boldsymbol{u} \in V$ that satisfies the equilibrium condition. Density ρ, gravitational acceleration $\boldsymbol{g}$ and displacement V are considered as a set of all solution candidates that satisfy the admissible function for the displacements, or the displacement boundary condition, in other words.

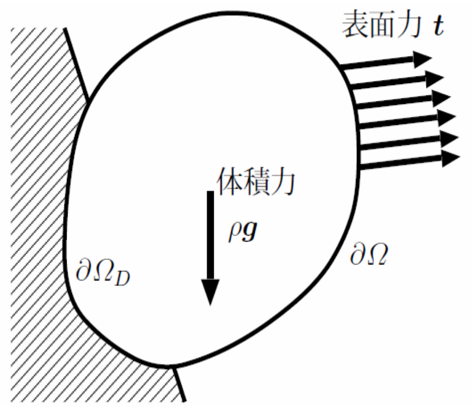


図 1: 境界値問題概念図

- Linear elastic body obeys the Hooke's law. The microscopic transformation of such substance, the iron and the rubber, for example, are commonly known as isotropic, and its internal stress all depend on the displacement. The substance can be made a model.
- Displacement boundary condition or the surface force are given at all points on the surface of substance $\partial\Omega$. Which implies the surface force is being provided at all points but $\partial\Omega D$. It is often omitted in a case in which the boundary value takes 0, therefore should be carefully observed.

Definitions of Symbols

- We define a configuration of the substance at nominal time t_0 as a nominal configuration, and express the position vector at each substance point as $\boldsymbol{X}$
- Position vector of a mass point $\boldsymbol{X}$ at the present time t is expressed as $\boldsymbol{x}$
- Displacement vector for the substance point from t0 to t is expressed as $\boldsymbol{u}$

$$\boldsymbol{u} = \boldsymbol{x} - \boldsymbol{X} \tag{1}$$

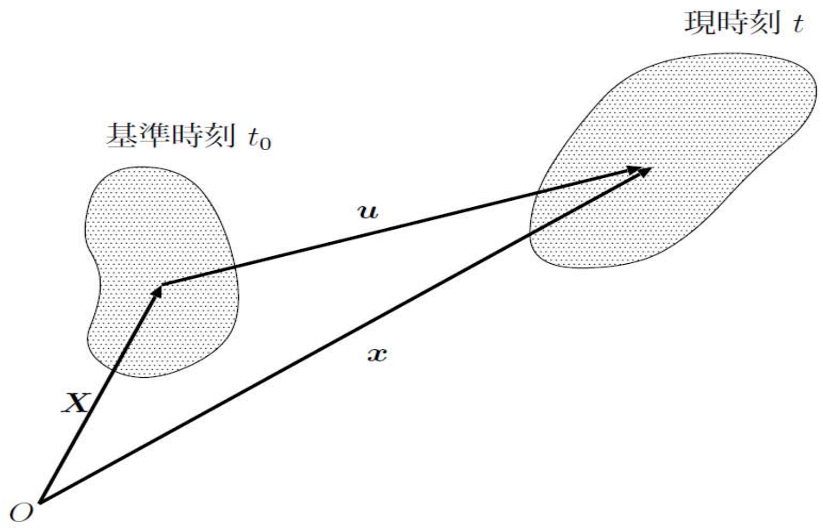


図 2: 物質点の運動

Strong Formulation 1

This problems can be formulated by the following.
[B] Where $\boldsymbol{t}, \boldsymbol{g}$ are given, find $\boldsymbol{u} \in V$ that satisfies the following:
[1] Balance equation(Cauchy' equation of motion)

$$\nabla_x \cdot \boldsymbol{T} + \rho \boldsymbol{g} = 0 \tag{2}$$

[2] Boundary condition equation

$$\boldsymbol{u} = \underline{\boldsymbol{u}} \quad \text{on} \quad \partial\Omega_D \tag{3}$$

$$\boldsymbol{T}^T \cdot \boldsymbol{n} = \boldsymbol{t} \quad \text{on} \quad \partial\Omega - \partial\Omega_D \tag{4}$$

[3] Displacement・strain relational expression

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{5}$$

[4] Stress・strain relational expression(constructive equation)

$$T_{ij} = \kappa(\operatorname{div} \boldsymbol{u})\delta_{ij} + 2G\varepsilon^D_{ij}(\boldsymbol{u}) \tag{6}$$

- In any problems, [1] and [2] are congruent. (possibly reformed in equivalent expressions if necessary.) [4] depends on its substance model, and [3] is determined in correspond to [4]

Definition of Symbols

This problem can be formulated as in the following:

[*B*] With given $\boldsymbol{t}$ and $\boldsymbol{g}$, obtain $\boldsymbol{u} \in V$ that satisfies the following equations.

$$\nabla_x \cdot \boldsymbol{T} + \rho \boldsymbol{g} = 0 \tag{7}$$

$$\boldsymbol{T}^T \cdot \boldsymbol{n} = \boldsymbol{t} \tag{8}$$

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{9}$$

$$T_{ij} = \kappa(\operatorname{div}\boldsymbol{u})\delta_{ij} + 2G\varepsilon_{ij}^D(\boldsymbol{u}) \tag{10}$$

- A set of all admissible function of the displacement V
- $\boldsymbol{T}$ Cauchy stress
- κ, G bulk modulus, modulus of rigidity (physical property)
- δ_{ij} Kronecker delta sym

$$\delta_{ij} = \begin{cases} 1 & (i = j) \\ 0 & (i \neq j) \end{cases} \tag{11}$$

- ε_{ij}, ε_{ij}^D linear strain, deviator strain

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{12}$$

$$\varepsilon_{ij}^D(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) - \frac{1}{3}(\operatorname{div}\boldsymbol{u})\delta_{ij}, \quad \operatorname{div}\boldsymbol{u} = \frac{\partial u_i}{\partial X_i} = \frac{\partial u_1}{\partial X_1} + \frac{\partial u_2}{\partial X_2} + \frac{\partial u_3}{\partial X_3} = \operatorname{tr}(\varepsilon_{ij}) \tag{13}$$

Weak Formulation

- As we stated earlier, the finite element method is associated with the approximate analysis of the weak form of the differential equations.
- [V] represents the weak form corresponding to [B].

[V] With the surface force $\boldsymbol{t}$ and the body force $\rho\boldsymbol{g}$ given, obtain $\boldsymbol{u} \in V$ that satisfies the following.

$$\int_{\Omega} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega = \int_{\partial\Omega} \delta\boldsymbol{u}\cdot\boldsymbol{t}\,\mathrm{d}S + \int_{\Omega} \rho\,\delta\boldsymbol{u}\cdot\boldsymbol{g}\,\mathrm{d}\Omega \qquad \forall\delta\boldsymbol{u} \in \mathcal{V} \tag{14}$$

- summation convention is used for $Tij(\boldsymbol{v})\ \varepsilon ij(\delta\boldsymbol{u})$
- Therefore,

$$\begin{aligned} T_{ij}(\boldsymbol{v})\,\varepsilon_{ij}(\boldsymbol{v}) &= \sum_{i=1}^{3}\sum_{j=1}^{3} T_{ij}(\boldsymbol{v})\,\varepsilon_{ij}(\boldsymbol{v}) \\ = \quad & T_{11}(\boldsymbol{v})\,\varepsilon_{11}(\boldsymbol{v}) + T_{12}(\boldsymbol{v})\,\varepsilon_{12}(\boldsymbol{v}) + T_{13}(\boldsymbol{v})\,\varepsilon_{13}(\boldsymbol{v}) \\ & + T_{21}(\boldsymbol{v})\,\varepsilon_{21}(\boldsymbol{v}) + T_{22}(\boldsymbol{v})\,\varepsilon_{22}(\boldsymbol{v}) + T_{23}(\boldsymbol{v})\,\varepsilon_{23}(\boldsymbol{v}) \\ & + T_{31}(\boldsymbol{v})\,\varepsilon_{31}(\boldsymbol{v}) + T_{32}(\boldsymbol{v})\,\varepsilon_{32}(\boldsymbol{v}) + T_{33}(\boldsymbol{v})\,\varepsilon_{33}(\boldsymbol{v}) \end{aligned} \tag{15}$$

Discretization and Dividing Finite Element 1

- In the finite element method, the region Ω , the analysis object is divided in the elements with the finite magnitude. Which is expressed in the following formulation,

$$\Omega = \sum_e \Omega_e \tag{16}$$

- Therefore, the regional integration along with the boundary integration may be gained by:

$$\int_{\Omega} \mathrm{d}\Omega = \sum_e \int_{\Omega_e} \mathrm{d}\Omega \tag{17}$$

$$\int_{\partial\Omega} \mathrm{d}S = \sum_e \int_{\partial\Omega_e} \mathrm{d}S \tag{18}$$

- Thus, the

$$\sum_e \left[\int_{\Omega_e} T_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\delta\boldsymbol{u})\, \mathrm{d}\Omega \right.$$
$$\left. - \int_{\partial\Omega_e} \delta\boldsymbol{u} \cdot \boldsymbol{t}\, \mathrm{d}S - \int_{\Omega_e} \rho\, \delta\boldsymbol{u} \cdot \boldsymbol{g}\, \mathrm{d}\Omega \right] = 0 \qquad \forall \delta\boldsymbol{u} \in \mathcal{V} \tag{19}$$

We assume ***x*** and ***u***, whic tions within each element.

$$x_i = N^{(j)} x_i^{(j)} \tag{20}$$

$$u_i = N^{(j)} u_i^{(j)} \tag{21}$$

Matrix Notation

- We utilize the matrix notations for the convenience in the calculations.
- The matrix notations we show in the following are fundamentally introduced as a procedural means, and which contains no intrinsic implications, therefore, each programmer may arrange his/her own way to meet the needs.
- We introduce the most common and applicable procedures in the following.

Stress-Strain Matrix([D] Matrix) 1

- [Ve] The integrands $Tij\,(\boldsymbol{u})\ \varepsilon ij(\,\delta \boldsymbol{u})$ in the left hand side in the first term may be expressed as the following if the summation convention was not being used.

$$\sum_e \left[\int_{\Omega_e} T_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\delta \boldsymbol{u})\, \mathrm{d}\Omega \right.$$
$$\left. - \int_{\partial\Omega_e} \delta \boldsymbol{u} \cdot \boldsymbol{t}\, \mathrm{d}S - \int_{\Omega_e} \rho\, \delta \boldsymbol{u} \cdot \boldsymbol{g}\, \mathrm{d}\Omega \right] = 0 \qquad \forall \delta \boldsymbol{u} \in \mathcal{V} \tag{22}$$

$$\begin{aligned} T_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\delta \boldsymbol{u}) = \quad & T_{11}(\boldsymbol{u})\, \varepsilon_{11}(\delta \boldsymbol{u}) + T_{12}(\boldsymbol{u})\, \varepsilon_{12}(\delta \boldsymbol{u}) + T_{13}(\boldsymbol{u})\, \varepsilon_{13}(\delta \boldsymbol{u}) \\ & + T_{21}(\boldsymbol{u})\, \varepsilon_{21}(\delta \boldsymbol{u}) + T_{22}(\boldsymbol{u})\, \varepsilon_{22}(\delta \boldsymbol{u}) + T_{23}(\boldsymbol{u})\, \varepsilon_{23}(\delta \boldsymbol{u}) \\ & + T_{31}(\boldsymbol{u})\, \varepsilon_{31}(\delta \boldsymbol{u}) + T_{32}(\boldsymbol{u})\, \varepsilon_{32}(\delta \boldsymbol{u}) + T_{33}(\boldsymbol{u})\, \varepsilon_{33}(\delta \boldsymbol{u}) \end{aligned} \tag{23}$$

- Using the symmetry property of Tij and εij about I and j, organize the equations in order to have the least operation times

$$\begin{aligned} & T_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\delta \boldsymbol{u}) \\ & = \quad T_{11}(\boldsymbol{u})\, \varepsilon_{11}(\delta \boldsymbol{u}) + \ T_{22}(\boldsymbol{u})\, \varepsilon_{22}(\delta \boldsymbol{u}) + \ T_{33}(\boldsymbol{u})\, \varepsilon_{33}(\delta \boldsymbol{u}) \\ & + 2T_{12}(\boldsymbol{u})\, \varepsilon_{12}(\delta \boldsymbol{u}) + 2T_{23}(\boldsymbol{u})\, \varepsilon_{23}(\delta \boldsymbol{u}) + 2T_{31}(\boldsymbol{u})\, \varepsilon_{31}(\delta \boldsymbol{u}) \\ & = \{\varepsilon(\delta \boldsymbol{u})\}^T \{T(\boldsymbol{u})\} \end{aligned} \tag{24}$$

- $\{\varepsilon(\boldsymbol{v})\}$, $\{T(\boldsymbol{v})\}$ is defined by the following equations.

$$\{\varepsilon(\boldsymbol{v})\} = \begin{Bmatrix} \varepsilon_{11}(\boldsymbol{v}) \\ \varepsilon_{22}(\boldsymbol{v}) \\ \varepsilon_{33}(\boldsymbol{v}) \\ 2\varepsilon_{12}(\boldsymbol{v}) \\ 2\varepsilon_{23}(\boldsymbol{v}) \\ 2\varepsilon_{31}(\boldsymbol{v}) \end{Bmatrix}, \quad \{T(\boldsymbol{v})\} = \begin{Bmatrix} T_{11}(\boldsymbol{v}) \\ T_{22}(\boldsymbol{v}) \\ T_{33}(\boldsymbol{v}) \\ T_{12}(\boldsymbol{v}) \\ T_{23}(\boldsymbol{v}) \\ T_{31}(\boldsymbol{v}) \end{Bmatrix} \tag{25}$$

Stress-Strain Matrix([D] Matrix) 2

- Relational expression for the stress Tij and the strain εij can be,

$$T_{ij} = \kappa(\operatorname{div} \boldsymbol{u})\delta_{ij} + 2G\varepsilon_{ij}^{D}(\boldsymbol{u}) \tag{26}$$

Based on the relational expression, have $\{T(\boldsymbol{v})\}$ and $\{\varepsilon(\boldsymbol{v})\}$ correlate with the matrix and the vector product formulations.

$$\{T(\boldsymbol{v})\} = [D]\{\varepsilon(\boldsymbol{v})\} \tag{27}$$

- This matrix [D] is often called the stress-strain matrix, or simply called [D] matrix.
- We can write out the components of Tij found in $\{Tij(\boldsymbol{v})\}$,

$$T_{11} = \kappa\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) + 2G\varepsilon_{11} - \frac{2G}{3}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) \tag{28}$$

$$T_{22} = \kappa\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) + 2G\varepsilon_{22} - \frac{2G}{3}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) \tag{29}$$

$$T_{33} = \kappa\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) + 2G\varepsilon_{33} - \frac{2G}{3}\left(\varepsilon_{11} + \varepsilon_{22} + \varepsilon_{33}\right) \tag{30}$$

$$T_{12} = 2G\varepsilon_{12} \tag{31}$$

$$T_{23} = 2G\varepsilon_{23} \tag{32}$$

$$T_{31} = 2G\varepsilon_{31} \tag{33}$$

Stress-Strain Matrix([D] Matrix) 3

- It might look a little pressing to bring then into the matrix expressions though, we obtain the following.

$$\begin{Bmatrix} T_{11} \\ T_{22} \\ T_{33} \\ T_{12} \\ T_{23} \\ T_{31} \end{Bmatrix} = \begin{bmatrix} \kappa & \kappa & \kappa & & & \\ \kappa & \kappa & \kappa & & 0 & \\ \kappa & \kappa & \kappa & & & \\ & & & & & \\ & 0 & & & & \\ & & & & & \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{Bmatrix} + \begin{bmatrix} \frac{4}{3}G & -\frac{2}{3}G & -\frac{2}{3}G & & & \\ -\frac{2}{3}G & \frac{4}{3}G & -\frac{2}{3}G & & 0 & \\ -\frac{2}{3}G & -\frac{2}{3}G & \frac{4}{3}G & & & \\ & & & G & & \\ & 0 & & & G & \\ & & & & & G \end{bmatrix} \begin{Bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ 2\varepsilon_{12} \\ 2\varepsilon_{23} \\ 2\varepsilon_{31} \end{Bmatrix} \tag{34}$$

- Now we define [Dv], [Dd] in the next step.

$$[D_v] = \begin{bmatrix} \kappa & \kappa & \kappa & & & \\ \kappa & \kappa & \kappa & & 0 & \\ \kappa & \kappa & \kappa & & & \\ & & & & & \\ & 0 & & & & \\ & & & & & \end{bmatrix} \quad [D_d] = \begin{bmatrix} \frac{4}{3}G & -\frac{2}{3}G & -\frac{2}{3}G & & & \\ -\frac{2}{3}G & \frac{4}{3}G & -\frac{2}{3}G & & 0 & \\ -\frac{2}{3}G & -\frac{2}{3}G & \frac{4}{3}G & & & \\ & & & G & & \\ & 0 & & & G & \\ & & & & & G \end{bmatrix} \tag{35}$$

- Using the matrix notation obtained in above, [D] is defined by

$$[D] = [D_v] + [D_d] \tag{36}$$

- . Furthermore, the integrands $T_{ij}(\boldsymbol{u})\ \varepsilon_{ij}(\delta\boldsymbol{u})$ found on the left hand side in the first term [Ve] can be expressed as

$$T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) = \{\varepsilon(\delta\boldsymbol{u})\}^T\,\{\boldsymbol{T}(\boldsymbol{u})\}$$
$$= \{\varepsilon(\delta\boldsymbol{u})\}^T\,[D]\,\{\varepsilon(\boldsymbol{u})\} \tag{37}$$

Node Displacement-Strain Matrix($[B]$ Matrix) 1

- Displacement and linear strain

$$\varepsilon_{ij}(\boldsymbol{u}) = \frac{1}{2}\left(\frac{\partial u_i}{\partial X_j} + \frac{\partial u_j}{\partial X_i}\right) \tag{38}$$

- Displacement and the node displacement

$$u_i = N^{(j)} u_i^{(j)} \tag{39}$$

- Collecting all together, the linear strains and the node displacements are correlated with the following matrix and vector product formulations
- This matrix $[B]$ is called the node displacement-strain matrix, or simply $[B]$ matrix. n represents the number of the nodes found in the single element.

$$\{\varepsilon(\boldsymbol{u})\} = [B]\{u_i^{(n)}\} \tag{40}$$

- $\{u(n)i\}$ is defined by the following equation.

$$\{u_i^{(n)}\} = \left\{u_1^{(1)}, u_2^{(1)}, u_3^{(1)}, u_1^{(2)}, u_2^{(2)}, u_3^{(2)}, \cdots, u_1^{(n)}, u_2^{(n)}, u_3^{(n)}\right\}^T \tag{41}$$

Node Displacement-Strain Matrix([B] Matrix) 2

- Since $\frac{\partial u_i}{\partial X_j}$ needed in the calculation of the strain represents the quantity of which the node displacement does not depend on the position vector ***x*****,** we can write as,

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial N^{(n)}}{\partial X_j} u_i^{(n)} \tag{42}$$

- Moreover,

$$\varepsilon_{11} = \frac{\partial u_1}{\partial X_1} \tag{43}$$

$$\varepsilon_{22} = \frac{\partial u_2}{\partial X_2} \tag{44}$$

$$\varepsilon_{33} = \frac{\partial u_3}{\partial X_3} \tag{45}$$

$$2\varepsilon_{12} = \frac{\partial u_1}{\partial X_2} + \frac{\partial u_2}{\partial X_1} \tag{46}$$

$$2\varepsilon_{23} = \frac{\partial u_2}{\partial X_3} + \frac{\partial u_3}{\partial X_2} \tag{47}$$

$$2\varepsilon_{31} = \frac{\partial u_3}{\partial X_1} + \frac{\partial u_1}{\partial X_3} \tag{48}$$

In considering the above,

Node Displacement-Strain Matrix($[B]$ Matrix) 3

- Specifically, the components are,

$$
\begin{aligned}
\varepsilon_{11} &= \frac{\partial N^{(1)}}{\partial X_1} u_1^{(1)} + \frac{\partial N^{(2)}}{\partial X_1} u_1^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_1} u_1^{(n)} \\
\varepsilon_{22} &= \frac{\partial N^{(1)}}{\partial X_2} u_2^{(1)} + \frac{\partial N^{(2)}}{\partial X_2} u_2^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_2} u_2^{(n)} \\
\varepsilon_{33} &= \frac{\partial N^{(1)}}{\partial X_3} u_3^{(1)} + \frac{\partial N^{(2)}}{\partial X_3} u_3^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_3} u_3^{(n)} \\
2\varepsilon_{12} &= \frac{\partial N^{(1)}}{\partial X_1} u_2^{(1)} + \frac{\partial N^{(2)}}{\partial X_1} u_2^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_1} u_2^{(n)} \\
&\quad + \frac{\partial N^{(1)}}{\partial X_2} u_1^{(1)} + \frac{\partial N^{(2)}}{\partial X_2} u_1^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_2} u_1^{(n)} \\
2\varepsilon_{23} &= \frac{\partial N^{(1)}}{\partial X_2} u_3^{(1)} + \frac{\partial N^{(2)}}{\partial X_2} u_3^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_2} u_3^{(n)} \\
&\quad + \frac{\partial N^{(1)}}{\partial X_3} u_2^{(1)} + \frac{\partial N^{(2)}}{\partial X_3} u_2^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_3} u_2^{(n)} \\
2\varepsilon_{31} &= \frac{\partial N^{(1)}}{\partial X_3} u_1^{(1)} + \frac{\partial N^{(2)}}{\partial X_3} u_1^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_3} u_1^{(n)} \\
&\quad + \frac{\partial N^{(1)}}{\partial X_1} u_3^{(1)} + \frac{\partial N^{(2)}}{\partial X_1} u_3^{(2)} + \cdots + \frac{\partial N^{(n)}}{\partial X_1} u_3^{(n)}
\end{aligned}
\tag{49}
$$

•Based on the components studied in the previous, $[B]$ matrix can be represented in the 6×3 submatrix $[B^{(k)}]$.

$$[B^{(k)}] = \begin{bmatrix} \frac{\partial N^{(k)}}{\partial X_1} & & \\ & \frac{\partial N^{(k)}}{\partial X_2} & \\ & & \frac{\partial N^{(k)}}{\partial X_3} \\ \frac{\partial N^{(k)}}{\partial X_2} & \frac{\partial N^{(k)}}{\partial X_1} & \\ & \frac{\partial N^{(k)}}{\partial X_3} & \frac{\partial N^{(k)}}{\partial X_2} \\ \frac{\partial N^{(k)}}{\partial X_3} & & \frac{\partial N^{(k)}}{\partial X_1} \end{bmatrix} \tag{50}$$

$$[B] = [[B^{(1)}], [B^{(2)}], \ldots, [B^{(n)}]] \tag{51}$$

Element Stiffness Matrix

- By using [B], the integrands $Tij(\boldsymbol{u})\ \varepsilon ij(\delta\boldsymbol{u})$ found in the first term in [Ve] may be expressed by,

$$\begin{aligned} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u}) &= \{\varepsilon(\delta\boldsymbol{u})\}^T\,\{\boldsymbol{T}(\boldsymbol{u})\} \\ &= \{\varepsilon(\delta\boldsymbol{u})\}^T\,[D]\,\{\varepsilon(\boldsymbol{u})\} \\ &= \{\delta u_i^{(n)}\}^T\,[B]^T\,[D]\,[B]\,\{u_i^{(n)}\} \end{aligned} \tag{52}$$

- $\{\delta u_i^{(n)}\}$, $\{u_i^{(n)}\}$ are the values at the nodal points, and which do not depend on the regional integration because they become constant under the region, thus, we may take them out from the integrals.

$$\begin{aligned} &\int_{\Omega_e} T_{ij}(\boldsymbol{u})\,\varepsilon_{ij}(\delta\boldsymbol{u})\,\mathrm{d}\Omega \\ &= \int_{\Omega_e} \{\delta u_i^{(n)}\}^T\,[B]^T\,[D]\,[B]\,\{u_i^{(n)}\}\,\mathrm{d}\Omega \\ &= \{\delta u_i^{(n)}\}^T \left[\int_{\Omega_e} [B]^T\,[D]\,[B]\,\mathrm{d}\Omega\right] \{u_i^{(n)}\} \end{aligned} \tag{53}$$

- This integrated matrix is called the element stiffness matrix.

$$[K^{(e)}] = \int_{\Omega_e} [B]^T\,[D]\,[B]\,\mathrm{d}\Omega \tag{54}$$

External Force Vector

- For the second and third terms in the left hand side [*Ve*], we prepare for the vectors in the node displacements to have them singled out from the integrals.

$$\sum_e \left[\int_{\Omega_e} T_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\delta \boldsymbol{u})\, \mathrm{d}\Omega - \int_{\partial\Omega_e} \delta \boldsymbol{u} \cdot \boldsymbol{t}\, \mathrm{d}S - \int_{\Omega_e} \rho\, \delta \boldsymbol{u} \cdot \boldsymbol{g}\, \mathrm{d}\Omega \right] = 0 \qquad \forall \delta \boldsymbol{u} \in \mathcal{V} \tag{55}$$

$$\begin{aligned} \int_{\partial\Omega_e} \delta u_i \cdot t_i\, \mathrm{d}S &= \int_{\partial\Omega_e} \{\delta u_i^{(n)}\}^T [N]^T \{t\}\, \mathrm{d}S \\ &= \{\delta u_i^{(n)}\}^T \int_{\partial\Omega_e} [N]^T \{t\}\, \mathrm{d}S \end{aligned} \tag{56}$$

$$\begin{aligned} \int_{\Omega_e} \rho\, \delta u_i \cdot g_i\, \mathrm{d}\Omega &= \int_{\Omega_e} \rho\, \{\delta u_i^{(n)}\}^T [N]^T \{g\}\, \mathrm{d}\Omega \\ &= \{\delta u_i^{(n)}\}^T \int_{\Omega_e} \rho\, [N]^T \{g\}\, \mathrm{d}\Omega \end{aligned} \tag{57}$$

- Provided that,

$$[N] = \begin{bmatrix} N^{(1)} & & & N^{(2)} & & & & N^{(n)} & & \\ & N^{(1)} & & & N^{(2)} & & \cdots & & N^{(n)} & \\ & & N^{(1)} & & & N^{(2)} & & & & N^{(n)} \end{bmatrix} \tag{58}$$

$$\{t\} = \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix}, \quad \{g\} = \begin{Bmatrix} g_1 \\ g_2 \\ g_3 \end{Bmatrix} \tag{59}$$

- Based on above, the external force vector $\{F^{(e)}\}$ is defined as following,

$$\{F^{(e)}\} = \int_{\partial\Omega_e} [N]^T \{t\}\, \mathrm{d}S + \int_{\Omega_e} \rho\, [N]^T \{g\}\, \mathrm{d}\Omega \tag{60}$$

Total Stiffness Matrix 1

- To put in order,

$$\sum_e \left[\int_{\Omega_e} T_{ij}(\boldsymbol{u})\, \varepsilon_{ij}(\delta \boldsymbol{u})\, \mathrm{d}\Omega - \int_{\partial \Omega_e} \delta \boldsymbol{u} \cdot \boldsymbol{t}\, \mathrm{d}S - \int_{\Omega_e} \rho\, \delta \boldsymbol{u} \cdot \boldsymbol{g}\, \mathrm{d}\Omega \right] = 0 \qquad \forall \delta \boldsymbol{u} \in \mathcal{V} \tag{61}$$

Which can be modified by,

$$\sum_e \left[\{\delta u_i^{(n)}\}^T \left([K^{(e)}]\, \{u_i^{(n)}\} - \{F^{(e)}\} \right) \right] = 0 \tag{62}$$

- Without touching the left hand side, modify $\{\delta u_i^{(n)}\}, \{u_i^{(n)}\}$ to the forms, in which the nodal point numbers are provided out of the total numbers instead by the numbers of each element.

$$\{\delta u_1, \delta u_2, \cdots, \delta u_n\} \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & & \dots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = \{\delta u_1, \delta u_2, \cdots, \delta u_n\} \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} \tag{63}$$

- Unifying the both equations then yield the following,

$$\{\delta u_1, \delta u_2, \cdots, \delta u_n\} \left\{ \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & & \dots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} \right\} = \{0\} \tag{64}$$

Total Stiffness Matrix 2

- In order for the equation to form with the arbitrary $\delta \boldsymbol{u}$,

$$\{\delta u_1, \delta u_2, \cdots, \delta u_n\} \left\{ \begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & & \dots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} \right\} = \{0\} \tag{65}$$

- The following equation must be established.

$$\begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & & \dots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} - \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} = \{0\} \tag{66}$$

- Thus, the solutions obtained from the following system of linear equations should be the approximate solutions.

$$\begin{bmatrix} K_{11} & K_{12} & \dots & K_{1n} \\ K_{21} & & & \vdots \\ \vdots & & & \vdots \\ K_{n1} & & \dots & K_{nn} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{Bmatrix} = \begin{Bmatrix} F_1 \\ F_2 \\ \vdots \\ F_n \end{Bmatrix} \tag{67}$$

- In contructual analysis, this equations are often called the stiffness equations, and its matrix is called the total stiffness matrix.

Numerical Integration

- It is necessary to conduct either volume or area integration in obtaining the matrix.
- However, it is almost impossible to analytically conduct integration because the integrand becomes complicated.
- Thus we conduct numerical integration instead, and Newton-Coate integration along with Gauss integrations are among the most common methods.
- Both integrations approximate the integrand by Lagrange polynomials based on the characteristics of Lagrange polynomials to obtain integration numerically.

Lagrange Polynomials 1

- Approximate $f(x)$, $(a \le x \le b)$ by polynomials.
- Lagrange polynomials take the sampling points including both extremes of do $\{x_n\}, (a = x_1 < x_2 < \cdots < x_n = b)$ to be approximated by following,

$$f(x) \approx Q_n(x) = \sum_{k=1}^{n} f(x_k) H_k(x) \tag{68}$$

$$H_i(x) = \frac{(x - x_1)(x - x_2)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_1)(x_i - x_2)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)} \tag{69}$$

- H_k is $n-1$th order function that takes 1 at the sampling points, and 0 at any other points.

$$H_k(x_i) = \begin{cases} 1 & k = i \\ 0 & k \neq i \end{cases} \tag{70}$$

- Thus the sampling points x_k,

$$f(x_k) = Q_n(x_k) \tag{71}$$

- $Q_n(x)$ is $n-1$th order function, which coincides with $f(x)$ with n sampling points $i (i = 1, \cdots, n)$.
- For example, when $n = 2$ we have $x_1 = a$, $x_2 = b$

$$H_1(x) = \frac{x - b}{a - b}, \quad H_2(x) = \frac{x - a}{b - a} \tag{72}$$

$$\begin{aligned} f(x) \approx Q_n(x) &= \sum_{k=1}^{n} f(x_k) H_k(x) \\ &= f(a)\frac{x - b}{a - b} + f(b)\frac{x - a}{b - a} \end{aligned} \tag{73}$$

This represents a straight line connected by the end points.

- If we take $x_1 = -1$, $x_2 = 1$

$$H_1(x) = \frac{1}{2}(1 - x), \quad H_2(x) = \frac{1}{2}(1 + x) \tag{74}$$

Then we obtain the above, which coincide with the previous interpolation function in the single order.

Lagrange Polynomials 2

- Basic facts：When two nth- order polynomials$f(x)$, $g(x)$ coincide with another $n + 1$points $x_i(i = 1, \cdots,$ $n + 1)$, then *f* coincide with g, as well.
  Proof：Suppose we have $h(x) = f(x) - g(x)$ then $h(x)$ takes n th order polynomials. Now, under $x_i(i = 1, \cdots, n + 1)$, if $f(x)$ coincides with $g(x)$,

$$
\begin{aligned}
f(x_i) &= g(x_i) \quad (i = 1, \cdots, n+1) \\
h(x_i) &= f(x_i) - g(x_i) = 0 \quad (i = 1, \cdots, n+1) \\
&= a(x - x_1)(x - x_2) \cdots (x - x_{n+1}) = 0 \qquad (75)
\end{aligned}
$$

  Where *a* is an arbitrary coefficient. Hence, $h(x)$ becomes $n + 1$ th order function and there appears a contradiction.

- If we take $f(x)$ as n th order polynomials to approximate by Lagrange polynomials. For each $H_k(x)$ n th order function is taken with n+1 sampling points, $Q_{n+1}(x)$ becomes n *th* order function. Based on the facts, $f(x)$ and $Q_{n+1}(x)$ coincide regardless of how the sampling points are taken.

Basics to Numerical Integration

- Newton-Coate integration and Gauss integration are the method of numerically obtaining the integration based on the approximation by Lagrange polynomials.

$$\begin{aligned} \int_a^b f(x)dx &\approx \int_a^b Q_n(x)dx \\ &= \int_a^b \sum_{k=1}^{n} f(x_k)H_k(x)dx \\ &= \sum_{k=1}^{n} f(x_k) \int_a^b H_k(x)dx \end{aligned} \tag{76}$$

- The following integration value is gained regardless of *f*(*x*), but rather gained based on the information of the sampling points, and which is called the heaviness corresponding to the sampling points *xk.*

$$w_k = \int_a^b H_k(x)dx \tag{77}$$

- Therefore, we can obtain the approximation by multiplying the heaviness, corresponding to the alue at sampling points *f*(*xk*) and the point *xk,* to the integration of *f*(*x*) then add them all together.

$$\int_a^b f(x)dx \approx \int_a^b Q_n(x)dx = \sum_{k=1}^{n} f(x_k)w_k \tag{78}$$

Since integrand is approximated by Lagrange polynomials, we gain more accuracy with greater the number of the sampling points.However when integrand is *n th* polynomials, the solution coincides with analytical integration by taking the *n* + 1sampling points. And we observe no difference by taking more than *n* + 2 sampling points.

- $x = \frac{a}{2}(1-r) + \frac{b}{2}(1+r)$ hen.

$$\int_a^b f(x)dx = \int_{-1}^1 f(x(r))\frac{dx}{dr}dr = \int_{-1}^1 f(x(r))\frac{b-a}{2}dr = \frac{b-a}{2}\int_{-1}^1 f(x(r))dr \tag{79}$$

Thus,

$$\int_a^b f(x)dx = \frac{b-a}{2}\int_{-1}^{1} f(x(r))dr \approx \frac{b-a}{2}\int_{-1}^{1} Q_n(r)dr$$
$$= \frac{b-a}{2}\int_{-1}^{1}\sum_{k=1}^{n} f(x_k(r))H_k(r)dr = \frac{b-a}{2}\sum_{k=1}^{n} f(x_k(r))\int_{-1}^{1} H_k(r)dx \tag{80}$$

Discussion follows with a set integration interval from -1 to 1.

Newton-Coate Integration 1

- In Newton-Coate integration, the end points are included in selecting equal intervals of n sampling points.
- For $n = 2$ it is commonly called the trapezoidal rule, while $n = 3$, it is called Simpson integration.
- For the Trapezoidal rule,

$$H_1(x) = \frac{1}{2}(1 - x), \quad H_2(x) = \frac{1}{2}(1 + x) \tag{81}$$

Therefore obtained by following,

$$\begin{aligned} w_1 &= \int_{-1}^{1} H_1(x)dx = \frac{1}{2}\int_{-1}^{1}(1 - x)dx = 1 \\ w_2 &= \int_{-1}^{1} H_2(x)dx = \frac{1}{2}\int_{-1}^{1}(1 + x)dx = 1 \end{aligned} \tag{82}$$

Newton-Coate Integration 2

- In Simpson integration,

$$H_1(x) = \frac{1}{2}x(x-1)$$
$$H_2(x) = 1 - x^2$$
$$H_3(x) = \frac{1}{2}x(x+1) \qquad (83)$$

Therefore obtained by following

$$w_1 = \int_{-1}^{1} H_1(x)dx = \int_{-1}^{1} \frac{1}{2}x(x-1)dx = \frac{1}{3}$$
$$w_2 = \int_{-1}^{1} H_2(x)dx = \int_{-1}^{1} 1 - x^2 dx = \frac{4}{3}$$
$$w_3 = \int_{-1}^{1} H_3(x)dx = \int_{-1}^{1} \frac{1}{2}x(x+1)dx = \frac{1}{3} \qquad (84)$$

- Obviously, when we obtain the integrand with $n-1$polynomials, the integrals can be achieved by taking more than n sampling points.
- In considering the odd function to have its integrals 0, $(2n-1)$ polynomials can be accurately obtained if $(2n-1)$ sampling points are taken.
- Thus in conducting Newton-Coate integration, often odd numbers of sampling points are taken.

Gauss Integration 1

- In Gauss integration, integrand is approximated by $(2n-1)$ order function.

$$f(x) \approx R_n(x) = \underbrace{\sum_{k=1}^{n} f(x_k) H_k(x)}_{Q_n(x)} + q(x) \sum_{k=1}^{n} a_k x^{k-1} \tag{85}$$

ak takes an arbitrary coefficient, and $q(x)$ expresses the following n polynomials.

$$q(x) = (x - x_1)(x - x_2) \cdots (x - x_n) \tag{86}$$

- At sampling point $Q_n(x_k) = f(k_k,), q(x_k) = 0$ thus,

$$f(x_k) = R_n(x_k) \tag{87}$$

- Here the position of sampling points $x_k (k = 1, \cdots, n)$ is expressed by

$$\int_{-1}^{1} q(x) x^{k-1} dx = 0 \tag{88}$$

In order to satisfy the above,

$$\begin{aligned}
\int_{-1}^{1} f(x)dx &\approx \int_{-1}^{1} R_n(x)dx \\
&= \int_{-1}^{1} \sum_{k=1}^{n} f(x_k)H_k(x)dx + \int_{-1}^{1} q(x) \sum_{k=1}^{n} a_k x^{k-1} dx \\
&= \int_{-1}^{1} \sum_{k=1}^{n} f(x_k)H_k(x)dx + \sum_{k=1}^{n} a_k \underbrace{\int_{-1}^{1} q(x)x^{k-1}dx}_{0} \\
&= \sum_{k=1}^{n} f(x_k) \int_{-1}^{1} H_k(x)dx \\
&= \sum_{k=1}^{n} w_k f(x_k) \qquad (89)
\end{aligned}$$

- Implying that integral of the integrand $f(x)$ is approximated as an integral of $2n-1$ th order function at n sampling points.

Gauss Integration 2

- Let us now find the specific positions for sampling points.
- When $n = 1$

$$\begin{aligned}\int_{-1}^{1}(x-x_1)x^{1-1}dx &= \int_{-1}^{1}-x_1dx \\ &= 2x_1 = 0 \qquad (90)\end{aligned}$$

From which to obtain the heaviness that corresponds to $x1 = 0$,

$$\begin{aligned}&h_1(x) = 1 \\ &\int_{-1}^{1}h_1(x)dx = \int_{-1}^{1}1dx = 2 \qquad (91)\end{aligned}$$

- When $n = 2$,

$$\begin{aligned}&\int_{-1}^{1}(x-x_1)(x-x_2)x^{1-1}dx = \frac{2}{3}+2x_1x_2 = 0 \\ &\int_{-1}^{1}(x-x_1)(x-x_2)x^{2-1}dx = -\frac{2}{3}(x_1+x_2) = 0 \qquad (92)\end{aligned}$$

- Then obtain weight corresponding to $x_1 = -\sqrt{\frac{1}{3}}, x_2 = \sqrt{\frac{1}{3}}$

$$h_1(x) = \frac{x - \sqrt{\frac{1}{3}}}{-\sqrt{\frac{1}{3}} - \sqrt{\frac{1}{3}}} = -\frac{\sqrt{3}}{2}\left(x - \sqrt{\frac{1}{3}}\right), \quad \int_{-1}^{1} h_1(x)dx = 1$$

$$h_2(x) = \frac{x + \sqrt{\frac{1}{3}}}{\sqrt{\frac{1}{3}} + \sqrt{\frac{1}{3}}} = \frac{\sqrt{3}}{2}\left(x + \sqrt{\frac{1}{3}}\right), \quad \int_{-1}^{1} h_2(x)dx = 1 \qquad (93)$$

Gauss Integration 3

- When n = 3,

$$
\begin{aligned}
&\int_{-1}^{1}(x-x_1)(x-x_2)(x-x_3)x^{1-1}dx \\
&= -\frac{2}{3}(x_1+x_2+x_3) - 2(x_1x_2x_3) = 0 \\
&\int_{-1}^{1}(x-x_1)(x-x_2)(x-x_3)x^{2-1}dx \\
&= \frac{2}{5} + \frac{2}{3}(x_1x_2 + x_2x_3 + x_3x_1) = 0 \\
&\int_{-1}^{1}(x-x_1)(x-x_2)(x-x_3)x^{3-1}dx \\
&= -\frac{2}{5}(x_1+x_2+x_3) - \frac{2}{3}(x_1x_2x_3) = 0
\end{aligned}
\qquad (94)
$$

- Find the weight that correlates with $x_1 = -\sqrt{\frac{3}{5}}, x_2 = 0, x_3 = \sqrt{\frac{3}{5}}$

$$
\begin{aligned}
h_1(x) &= \frac{x\left(x-\sqrt{\frac{3}{5}}\right)}{-\sqrt{\frac{3}{5}}2\sqrt{\frac{3}{5}}}, \quad \int_{-1}^{1} h_1(x)dx = \frac{5}{9} \\
h_2(x) &= \frac{\left(x-\sqrt{\frac{3}{5}}\right)\left(x+\sqrt{\frac{3}{5}}\right)}{-\sqrt{\frac{3}{5}}\sqrt{\frac{3}{5}}}, \quad \int_{-1}^{1} h_2(x)dx = \frac{8}{9} \\
h_3(x) &= \frac{x\left(x+\sqrt{\frac{3}{5}}\right)}{\sqrt{\frac{3}{5}}2\sqrt{\frac{3}{5}}}, \quad \int_{-1}^{1} h_3(x)dx = \frac{5}{9}
\end{aligned} \tag{95}
$$

Sampling Points in Actual Numerical Integration

- Obviously, the more we have the sampling points, the more accurate the solution we obtain.
- However, the more we have the sampling points, greater the amount of time spent on the calculation.
- Usually, in the first-order element, 2points taken by Gauss integration and 3points by Newton-Coate integration. In the second-order element, 3points used in Gauss integration and 5points used in Newton-Coate integration.

サンプリング点数	x_i	w_i
1	0	2
2	±0.57735 02691 89626	1
3	±0.77459 66692 41483	0.55555 55555 55556
	0	0.88888 88888 88889
4	±0.86113 63115 94053	0.34785 48451 37454
	±0.33998 10435 84856	0.65214 51548 62546
5	±0.90617 98459 38664	0.23692 68850 56189
	±0.53846 93101 05683	0.47862 86704 99366
	0	0.56888 88888 88889
6	±0.93246 95142 03152	0.17132 44923 79170
	±0.66120 93864 66265	0.36076 15730 48139
	±0.23861 91860 83197	0.46791 39345 72691

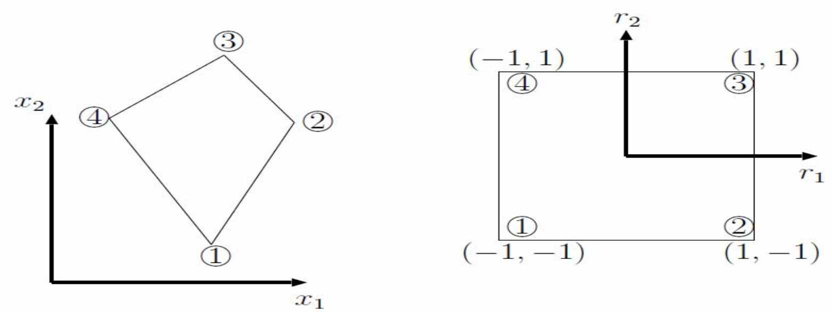
表 1: Gauss 積分のサンプリング点と重みの値

4 Noded Quadrilateral Solid Element

In one-dimensional space, divide the domain of integration into n-interval then conduct the coordinate transformation at each interval of x coordinates in a linear segment of line to $r(-1 \leq r \leq 1)$, using interpolation function.

$$N^{(1)} = \frac{1}{2}(1 - r), \quad N^{(2)} = \frac{1}{2}(1 + r) \tag{96}$$

- Now, what do we find under two-dimensional space?
- First, divide domain of integration by rectangular with its apexes at $(-1,-1)$, $(1,-1)$, $(1, 1)$,and $(-1, 1)$, then conduct coordinate transformation using two parameters r_1, r_2 $(-1 \leq r_1 \leq 1, -1 \leq r_2 \leq 1)$.
- Therefore, in physical coordinate systems, the nodal points under such configuration in the figure on the left is made to correlate with what it shows in the figures on the right. This implies that a tetrahedron in the physical coordinate system is being projected to a square in r_1- r_2 coordinate system.



(a) 物理座標系 (b) 自然座標系

Interpolation Function

•• Interpolation functions takes forms in the following,

$$N^{(1)} = \frac{1}{4}(1 - r_1)(1 - r_2) \tag{97}$$

$$N^{(2)} = \frac{1}{4}(1 + r_1)(1 - r_2) \tag{98}$$

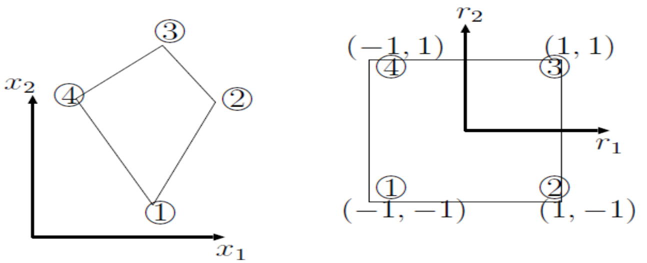
$$N^{(3)} = \frac{1}{4}(1 + r_1)(1 + r_2) \tag{99}$$

$$N^{(4)} = \frac{1}{4}(1 - r_1)(1 + r_2) \tag{100}$$

• In respect with one-dimensional space,

$$N^{(1)} = \frac{1}{2}(1 - r), \quad N^{(2)} = \frac{1}{2}(1 + r) \tag{101}$$

• Values at corresponding nodal points are found as 1, but in other nodal points, found as 0.



(a) 物理座標系　　(b) 自然座標系

Differentials in Discrete Value Expression 1

- Differentials of ui about xj, which are needed in calculating a strain, can be evaluated with chain rule in the following.

$$\frac{\partial u_i}{\partial x_j} = \frac{\partial N^{(n)}}{\partial x_j} u_i^{(n)} = \left(\frac{\partial N^{(n)}}{\partial r_1} \frac{\partial r_1}{\partial x_j} + \frac{\partial N^{(n)}}{\partial r_2} \frac{\partial r_2}{\partial x_j} \right) u_i^{(n)} \qquad (102)$$

- $\frac{\partial N^{(n)}}{\partial x_j}$ can be obtained also, with chain rule.

- Jacobian matrix $[J]$ may be found as

$$\begin{bmatrix} \frac{\partial N^{(n)}}{\partial r_1} \\ \frac{\partial N^{(n)}}{\partial r_2} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_2}{\partial r_1} \\ \frac{\partial x_1}{\partial r_2} & \frac{\partial x_2}{\partial r_2} \end{bmatrix} \begin{bmatrix} \frac{\partial N^{(n)}}{\partial x_1} \\ \frac{\partial N^{(n)}}{\partial x_2} \end{bmatrix} \qquad (103)$$

$$= [J] \begin{bmatrix} \frac{\partial N^{(n)}}{\partial x_1} \\ \frac{\partial N^{(n)}}{\partial x_2} \end{bmatrix} \qquad (104)$$

Differentials in Discrete Value Expression 2

- Each component of this Jacobian matrix $\frac{\partial x_i}{\partial r_j}$ is given by,

$$\frac{\partial x_i}{\partial r_j} = \frac{\partial N^{(n)}}{\partial r_j} x_i^{(n)} \tag{105}$$

- $\frac{\partial N^{(n)}}{\partial x_j}$ is evaluated as,

$$\begin{bmatrix} \frac{\partial N^{(n)}}{\partial x_1} \\ \frac{\partial N^{(n)}}{\partial x_2} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \frac{\partial N^{(n)}}{\partial r_1} \\ \frac{\partial N^{(n)}}{\partial r_2} \end{bmatrix} \tag{106}$$

- In addition, the regional integration can be expressed by,

$$\int_{\Omega_e} d\Omega = \int_{-1}^{1} \int_{-1}^{1} \det[J] dr_1 \, dr_2 \tag{107}$$

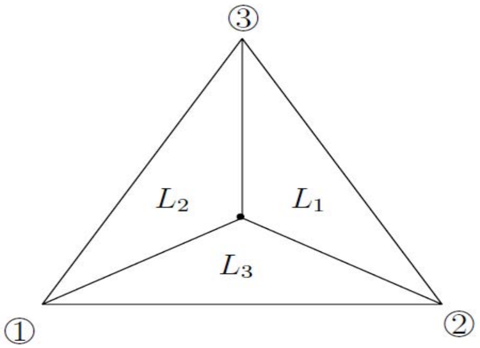
This integration is usually conducted by numerical integration method such as Gauss integration. Here, we use a doubled Gauss integration in one-dimensional space.

$$\int_{-1}^{1} \int_{-1}^{1} f(x, y) \, dx \, dy \approx \sum_i \sum_j w_i w_j f(x_i, y_j) \tag{108}$$

Interpolation Functions in Triangle Element 1

- Interpolation functions in triangle element are expressed in the area coordinates defined by the following.
- Area coordinates represent the coordinates consisted of the area of element A ,and the given points within the element. In addition, the area of triangles are given $A1$,$A2$ and $A3$(triangles made by the corresponding opposite sides of nodal points and its points)

$$
\begin{aligned}
L_1 &= A_1/A \\
L_2 &= A_2/A \\
L_3 &= A_3/A
\end{aligned}
\tag{109}
$$



$$L_1 + L_2 + L_3 = 1 \tag{110}$$

Interpolation Functions in Triangle Element 2

- Interpolation functions in single dimension with 3 nodal points

$$N^{(1)} = L_1 \tag{111}$$

$$N^{(2)} = L_2 \tag{112}$$

$$N^{(3)} = L_3 \tag{113}$$

- Interpolation functions in the two-dimensional 6 nodes,

$$N^{(1)} = L_1(2L_1 - 1) \tag{114}$$

$$N^{(2)} = L_2(2L_2 - 1) \tag{115}$$

$$N^{(3)} = L_3(2L_3 - 1) \tag{116}$$

$$N^{(4)} = 4L_2L_3 \tag{117}$$

$$N^{(5)} = 4L_3L_1 \tag{118}$$

$$N^{(6)} = 4L_1L_2 \tag{119}$$

- We can obtain the 6 nodes interpolation functions through 3 nodes functions.

Numerical Integration and Interpolation Functions in Triangular Element 1

- In actual calculations for element stiffness matrix, the numerical integration is necessary.
- Numerical integration is conducted by reflecting the area coordinates L_1, L_2, L_3 and the natural coordinates system r_1, r_2 in the way shows in the following.

$$r_1 = L_1 \tag{120}$$

$$r_2 = L_2 \tag{121}$$

$$1 - r_1 - r_2 = L_3 \tag{122}$$

- Domain for the triangle internal corresponds to the domain for the natural coordinates system appears in the figure below.

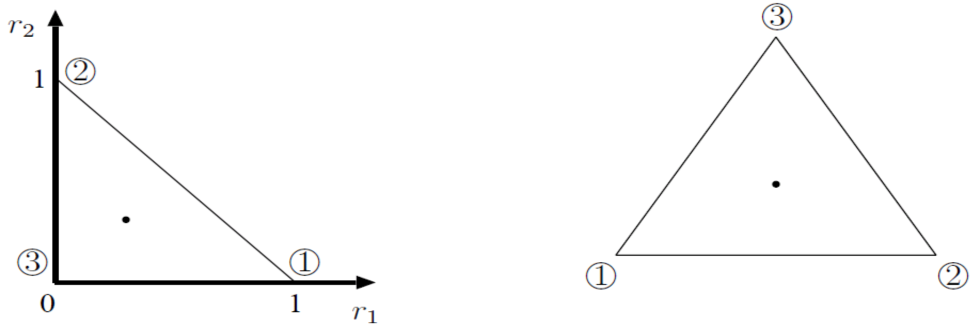


図 3: 三角形要素の自然座標と面積座標

Numerical Integration and Interpolation Functions in Triangular Element 2

- Under physical space, integral $\int_V dV$ transforms into the natural coordinates system by Jacobian matrix, in the same way we evaluated for the rectangular element.

$$\begin{bmatrix} \dfrac{\partial N^{(i)}}{\partial r_1} \\ \dfrac{\partial N^{(i)}}{\partial r_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x_1}{\partial r_1} & \dfrac{\partial x_2}{\partial r_1} \\ \dfrac{\partial x_1}{\partial r_2} & \dfrac{\partial x_2}{\partial r_2} \end{bmatrix}}_{\substack{\Downarrow \\ [J]}} \begin{bmatrix} \dfrac{\partial N^{(i)}}{\partial x_1} \\ \dfrac{\partial N^{(i)}}{\partial x_2} \end{bmatrix} \tag{123}$$

- Thus,

$$\begin{bmatrix} \dfrac{\partial N^{(i)}}{\partial x_1} \\ \dfrac{\partial N^{(i)}}{\partial x_2} \end{bmatrix} = [J^{-1}] \begin{bmatrix} \dfrac{\partial N^{(i)}}{\partial r_1} \\ \dfrac{\partial N^{(i)}}{\partial r_2} \end{bmatrix} \tag{124}$$

- Jacobian matrix component $\dfrac{\partial x_i}{\partial r_j}$ becomes what we obtained for the rectangular element in the following.

$$\frac{\partial x_i}{\partial r_j} = \frac{\partial N^{(n)}}{\partial r_j} x_i^{(n)} \tag{125}$$

Numerical Integration and Interpolation Functions in Triangular Element 3

- Here, a differential $\frac{\partial N^{(i)}}{\partial r_j}$ for shape functions by natural coordinates appears,

$$r_1 = L_1 \tag{126}$$

$$r_2 = L_2 \tag{127}$$

$$1 - r_1 - r_2 = L_3 \tag{128}$$

Based on the functions above, reflect with the area coordinates to obtain,

$$\frac{\partial N^{(i)}}{\partial r_1} = \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_1} \tag{129}$$

$$= \frac{\partial N^{(i)}}{\partial L_1} - \frac{\partial N^{(i)}}{\partial L_3} \tag{130}$$

$$\frac{\partial N^{(i)}}{\partial r_2} = \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_2} \tag{131}$$

$$= \frac{\partial N^{(i)}}{\partial L_2} - \frac{\partial N^{(i)}}{\partial L_3} \tag{132}$$

Numerical Integration and Interpolation Functions in Triangular Element 4

- In respect, conduct $\int_V dV \Rightarrow \int_0^1 \int_0^{1-r_1} \det J \, dr_2 \, dr_1$.

  Apparently in the form $\iint F \, dr_1 \, dr_2 = \frac{1}{2} \sum w_i F(x_i, y_i)$.

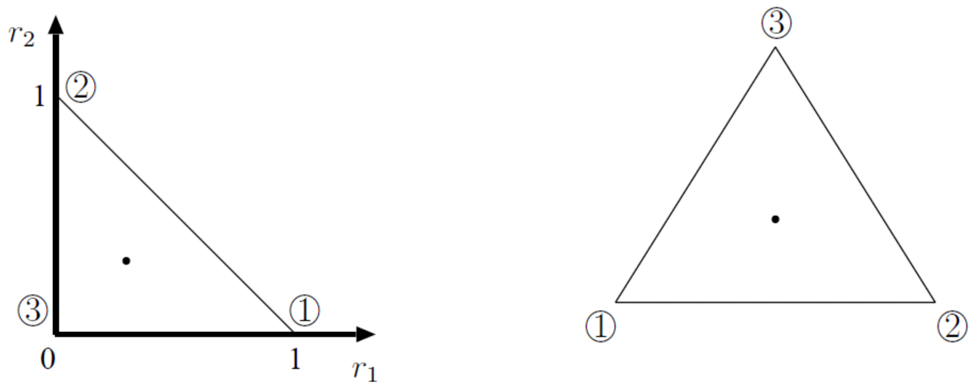


図 4: 三角形要素の自然座標と面積座標

Numerical Integration and Interpolation Functions in Triangular Element 5

Integration order	Degree of precision	r-coordinates	s-coordinates	Weights
3-point	2	$r_1 = 0.16666\,66666\,667$	$s_1 = r_1$	$w_1 = 0.33333\,33333\,333$
		$r_2 = 0.66666\,66666\,667$	$s_2 = r_1$	$w_2 = w_1$
		$r_3 = r_1$	$s_3 = r_2$	$w_3 = w_1$
7-point	5	$r_1 = 0.10128\,65073\,235$	$s_1 = r_1$	$w_1 = 0.12593\,91805\,448$
		$r_2 = 0.79742\,69853\,531$	$s_2 = r_1$	$w_2 = w_1$
		$r_3 = r_1$	$s_3 = r_2$	$w_3 = w_1$
		$r_4 = 0.47014\,20641\,051$	$s_4 = r_6$	$w_4 = 0.13239\,41527\,885$
		$r_5 = r_4$	$s_5 = r_4$	$w_5 = w_4$
		$r_6 = 0.05971\,58717\,898$	$s_6 = r_4$	$w_6 = w_4$
		$r_7 = 0.33333\,33333\,333$	$s_7 = r_7$	$w_7 = 0.225$
13-point	7	$r_1 = 0.06513\,01029\,002$	$s_1 = r_1$	$w_1 = 0.05334\,72356\,008$
		$r_2 = 0.86973\,97941\,956$	$s_2 = r_1$	$w_2 = w_1$
		$r_3 = r_1$	$s_3 = r_2$	$w_3 = w_1$
		$r_4 = 0.31286\,54960\,049$	$s_4 = r_6$	$w_4 = 0.07711\,37608\,903$
		$r_5 = 0.63844\,41885\,698$	$s_5 = r_4$	$w_5 = w_4$
		$r_6 = 0.04869\,03154\,253$	$s_6 = r_5$	$w_6 = w_4$
		$r_7 = r_5$	$s_7 = r_6$	$w_7 = w_4$
		$r_8 = r_4$	$s_8 = r_5$	$w_8 = w_4$
		$r_9 = r_6$	$s_9 = r_4$	$w_9 = w_4$
		$r_{10} = 0.26034\,59660\,790$	$s_{10} = r_{10}$	$w_{10} = 0.17561\,52574\,332$
		$r_{11} = 0.47930\,80678\,419$	$s_{11} = r_{10}$	$w_{11} = w_{10}$
		$r_{12} = r_{10}$	$s_{12} = r_{11}$	$w_{12} = w_{10}$
		$r_{13} = 0.33333\,33333\,333$	$s_{13} = r_{13}$	$w_{13} = -0.14957\,00444\,67$

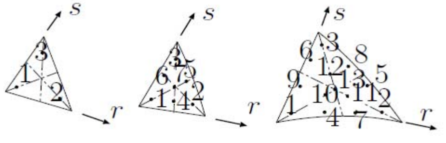
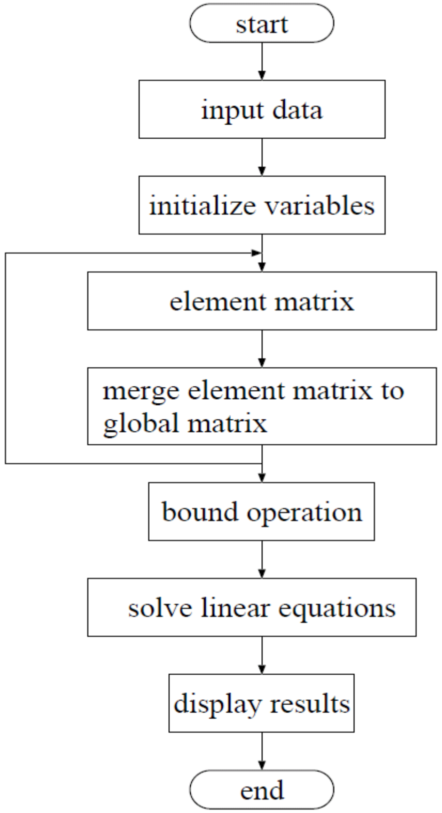


図 5: 積分点

Finite Element Analysis Code Prototype

• Finite element method programming structure can be,



• Basically, in linear finite method analysis coding, structure of the program stays the same. For dynamic analysis and the nonlinear analysis, the programs are based on this structure.

Input Data

- Maximum nodal points: MXNODE (1000)
- Maximum elements: MXELEM (1000)
- Maximum degree of freedom per 1 nodal point: MXDOFN (3)
- Maximum nodal points per 1 element: MXNOEL (8)
- Total nodal points: nnode
- Nodal points coordinates: coords(MXDOFN,MXNODE) 1
- Total elements: nelem
- Number of nodal points at each element: ntnoel(MXELEM)
- Connectivity: lnods(MXNOEL,MXELEM) 1
- Degrees of freedom per 1 nodal point: ndofn
- Total degree of freedom: ntotdf=nnode $\times$ndofn
- Number of nodal points at each element: ntnoel (MXELEM)

Drawing Element Stiffness Matrix

- To categorize the bugs occur in element stiffness programming,

1. Matrix$[D]$ and $[B]$
2. Jacobian Matrix
3. Numerical integration

A technique employed in verification of 2. and 3, is done by obtainng the volume of element in physical space and make a comparison with the actual volume.

- Volume of element in physical space can be gained by following,

$$\int_{\Omega_e} \mathrm{d}\Omega = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \det[J] dr_1 dr_2 dr_3 = \sum_{a=1}^{2}\sum_{b=1}^{2}\sum_{c=1}^{2} w_a\, w_b\, w_c \det[J(r_a, r_b, r_c)] \tag{133}$$

- Loop in triple numerical integration can be coded as,

$$\texttt{vol} = 0 \tag{134}$$
$$\textbf{for} \quad i1 = 1 \sim \texttt{nint} \tag{135}$$
$$\textbf{for} \quad i2 = 1 \sim \texttt{nint} \tag{136}$$
$$\textbf{for} \quad i3 = 1 \sim \texttt{nint} \tag{137}$$
$$r1 = (\text{sampring point } i1) \tag{138}$$
$$r2 = (\text{sampring point } i2) \tag{139}$$
$$r3 = (\text{sampring point } i3) \tag{140}$$
$$(\partial N / \partial r(r_1, r_2, r_3)) \tag{141}$$
$$(J, \det J) \tag{142}$$

–

$$w_1 = (\text{weight } i1) \tag{143}$$

$$w_2 = (\text{weight } i2) \tag{144}$$

$$w_3 = (\text{weight } i3) \tag{145}$$

$$\texttt{vol} = \texttt{vol} + \det J \cdot w1 \cdot w2 \cdot w3 \tag{146}$$

$$\textbf{end for} \tag{147}$$

$$\textbf{end for} \tag{148}$$

$$\textbf{end for} \tag{149}$$

Procedure of Drawing Stiffness Matrix– 4Noded Quadrilateral 1

- In finding element stiffness matrix for two-dimensional 4 noded quadrilateral element by 2 ×2 Gauss integration,
- Transform the domain of integration of the element stiffness matrix.（transformation of the domain of integration by isoparametric element)

$$[K^{(e)}] = \int_{\Omega_e} [B]^T [D] [B] \, \mathrm{d}\Omega$$

$$= \int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] \det[J] dr_2 dr_1 \tag{150}$$

$$[J] = \begin{bmatrix} \dfrac{\partial x_1}{\partial r_1} & \dfrac{\partial x_2}{\partial r_1} \\ \dfrac{\partial x_1}{\partial r_2} & \dfrac{\partial x_2}{\partial r_2} \end{bmatrix} \tag{151}$$

- Introduce the numerical integration to yield

$$\int_{-1}^{1} \int_{-1}^{1} [B]^T [D] [B] \det[J] dr_2 dr_1$$

$$= \sum_{a=1}^{2} \sum_{b=1}^{2} w_a \, w_b \, [B(r_a, r_b)]^T [D] [B(r_a, r_b)] \det[J(r_a, r_b)] \tag{152}$$

Procedure of Drawing Stiffness Matrix– 4Noded Quadrilateral 2

- In specific, components of Jacobian matrix can be evaluated by setting each sampling point r_a, r_b in the following,

$$
\begin{aligned}
\frac{\partial x_i}{\partial r_j}(r_a, r_b) &= \frac{\partial N^{(n)}}{\partial r_j}(r_a, r_b)\, x_i^{(n)} \\
&= \frac{\partial N^{(1)}}{\partial r_j}(r_a, r_b)\, x_i^{(1)} \\
&+ \frac{\partial N^{(2)}}{\partial r_j}(r_a, r_b)\, x_i^{(2)} \\
&+ \frac{\partial N^{(3)}}{\partial r_j}(r_a, r_b)\, x_i^{(3)} \\
&+ \frac{\partial N^{(4)}}{\partial r_j}(r_a, r_b)\, x_i^{(4)}
\end{aligned}
\tag{153}
$$

$$\frac{\partial N^{(1)}}{\partial r_1} = -\frac{1}{4}(1 - r_2) \tag{154}$$

$$\frac{\partial N^{(1)}}{\partial r_2} = -\frac{1}{4}(1 - r_1) \tag{155}$$

$$\frac{\partial N^{(2)}}{\partial r_1} = \frac{1}{4}(1 - r_2) \tag{156}$$

$$\frac{\partial N^{(2)}}{\partial r_2} = -\frac{1}{4}(1 + r_1) \tag{157}$$

$$\frac{\partial N^{(3)}}{\partial r_1} = \frac{1}{4}(1 + r_2) \tag{158}$$

$$\frac{\partial N^{(3)}}{\partial r_2} = \frac{1}{4}(1 + r_1) \tag{159}$$

$$\frac{\partial N^{(4)}}{\partial r_1} = -\frac{1}{4}(1 + r_2) \tag{160}$$

$$\frac{\partial N^{(4)}}{\partial r_2} = \frac{1}{4}(1 - r_1) \tag{161}$$

Clarify all components in the matrix, and again, draw out the Jacobian matrix, then follow through the steps to complete the calculation.

Procedure of Drawing Stiffness Matrix– 4Noded Quadrilateral 3

- [*B*] Matrix components are obtained by following

$$\begin{bmatrix} \dfrac{\partial N^{(n)}}{\partial x_1} \\ \dfrac{\partial N^{(n)}}{\partial x_2} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \dfrac{\partial N^{(n)}}{\partial r_1} \\ \dfrac{\partial N^{(n)}}{\partial r_2} \end{bmatrix} \tag{162}$$

- [*B*] Substitute each value into the corresponding part in matrix.

$$[B^{(k)}] = \begin{bmatrix} \dfrac{\partial N^{(k)}}{\partial x_1} & \\ & \dfrac{\partial N^{(k)}}{\partial x_2} \\ \dfrac{\partial N^{(k)}}{\partial x_2} & \dfrac{\partial N^{(k)}}{\partial x_1} \end{bmatrix} \tag{163}$$

$$[B] = \Big[[B^{(1)}], [B^{(2)}], \ldots, [B^{(n)}] \Big] \tag{164}$$

- From above, [*B*(*ra, rb*)] det[*J*(*ra, rb*)] is gained,

$$[B(r_a, r_b)]^T \, [D] \, [B(r_a, r_b)] \det[J(r_a, r_b)] \tag{165}$$

Calculate the above then multiply the weight *wa wb* then plug them into the configuration of the total stiffness matrix, $[K^{(e)}] = \int_{-1}^{1} \int_{-1}^{1} [B]^T \, [D] \, [B] \det[J] dr_1 dr_2$

$$= \sum_{a=1}^{2} \sum_{b=1}^{2} w_a \, w_b \, [B(r_a, r_b)]^T \, [D] \, [B(r_a, r_b)] \det[J(r_a, r_b)] \tag{166}$$

Procedure of Drawing Stiffness Matrix– Triangle 1

- For the triangle element, basically, we can take the same steps.
- Domain of integration in the element stiffness matrix is transformed as the figure indicates （transformation of integration domain by isoparametric element)

$$
\begin{aligned}
[K^{(e)}] &= \int_{\Omega_e} [B]^T [D] [B] \, \mathrm{d}\Omega \\
&= \int_0^1 \int_0^{1-r_1} [B]^T [D] [B] \det[J] dr_2 dr_1 \qquad (167)
\end{aligned}
$$

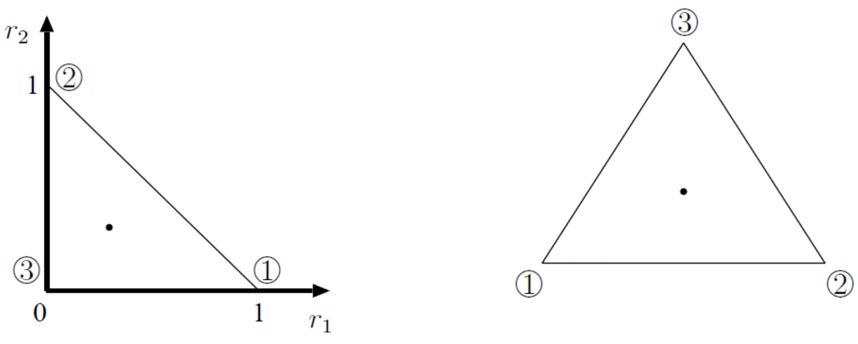


図 6: 三角形要素の自然座標と面積座標

$$[J] = \begin{bmatrix} \dfrac{\partial x_1}{\partial r_1} & \dfrac{\partial x_2}{\partial r_1} \\ \dfrac{\partial x_1}{\partial r_2} & \dfrac{\partial x_2}{\partial r_2} \end{bmatrix} \tag{168}$$

Procedure of Drawing Stiffness Matrix– Triangle 2

- Introduce the numerical integration

$$\int_0^1 \int_0^{1-r_1} [B]^T [D] [B] \det[J] dr_2 dr_1 = \frac{1}{2} \sum_{a,b=1}^{n} w_{ab} [B(r_a, r_b)]^T [D] [B(r_a, r_b)] \det[J(r_a, r_b)] \quad (169)$$

- Evaluate the values for each sampling point to draw the matrix, then plug them into the total stiffness matrix.

$$\int_0^1 \int_0^{1-r_1} [B]^T [D] [B] \det[J] dr_2 dr_1 = \frac{1}{2} \sum_{a.b=1}^{n} w_{ab} [B(r_a, r_b)]^T [D] [B(r_a, r_b)] \det[J(r_a, r_b)] \quad (170)$$

- Evaluate the values for each sampling point to draw the matrix, then plug them into the total stiffness matrix.

Procedure of Drawing Stiffness Matrix– Triangle 3

- Interpolation functions for 3 noded triangle element can be written by,

$$N^{(1)} = L_1 \tag{171}$$
$$N^{(2)} = L_2 \tag{172}$$
$$N^{(3)} = L_3 \tag{173}$$

- Base on this relations, reflect it to the area coordinates system in the following,

$$\frac{\partial N^{(i)}}{\partial r_1} = \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_1} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_1} \tag{174}$$
$$= \frac{\partial N^{(i)}}{\partial L_1} - \frac{\partial N^{(i)}}{\partial L_3} \tag{175}$$
$$\frac{\partial N^{(i)}}{\partial r_2} = \frac{\partial N^{(i)}}{\partial L_1}\frac{\partial L_1}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_2}\frac{\partial L_2}{\partial r_2} + \frac{\partial N^{(i)}}{\partial L_3}\frac{\partial L_3}{\partial r_2} \tag{176}$$
$$= \frac{\partial N^{(i)}}{\partial L_2} - \frac{\partial N^{(i)}}{\partial L_3} \tag{177}$$

- For the calculation detail,

$$\begin{array}{ccc} \dfrac{\partial N^{(1)}}{\partial r_1} = 1 & \dfrac{\partial N^{(2)}}{\partial r_1} = 0 & \dfrac{\partial N^{(3)}}{\partial r_1} = -1 \\ \dfrac{\partial N^{(1)}}{\partial r_2} = 0 & \dfrac{\partial N^{(2)}}{\partial r_2} = 1 & \dfrac{\partial N^{(3)}}{\partial r_2} = -1 \end{array} \tag{178}$$

Procedure of Drawing Stiffness Matrix– Triangle 4

- Jacobian matrix components, $\frac{\partial x_i}{\partial r_j}$ are,

$$\frac{\partial x_1}{\partial r_1} = \frac{\partial N^{(1)}}{\partial r_1} x_1^{(1)} + \frac{\partial N^{(2)}}{\partial r_1} x_1^{(2)} + \frac{\partial N^{(3)}}{\partial r_1} x_1^{(3)} = x_1^{(1)} - x_1^{(3)}$$
$$\frac{\partial x_2}{\partial r_1} = \frac{\partial N^{(1)}}{\partial r_1} x_2^{(1)} + \frac{\partial N^{(2)}}{\partial r_1} x_2^{(2)} + \frac{\partial N^{(3)}}{\partial r_1} x_2^{(3)} = x_2^{(1)} - x_2^{(3)}$$
$$\frac{\partial x_1}{\partial r_2} = \frac{\partial N^{(1)}}{\partial r_2} x_1^{(1)} + \frac{\partial N^{(2)}}{\partial r_2} x_1^{(2)} + \frac{\partial N^{(3)}}{\partial r_2} x_1^{(3)} = x_1^{(2)} - x_1^{(3)}$$
$$\frac{\partial x_2}{\partial r_2} = \frac{\partial N^{(1)}}{\partial r_2} x_2^{(1)} + \frac{\partial N^{(2)}}{\partial r_2} x_2^{(2)} + \frac{\partial N^{(3)}}{\partial r_2} x_2^{(3)} = x_2^{(2)} - x_2^{(3)} \qquad (179)$$

$$[J] = \begin{bmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_2}{\partial r_1} \\ \frac{\partial x_1}{\partial r_2} & \frac{\partial x_2}{\partial r_2} \end{bmatrix} = \begin{bmatrix} x_1^{(1)} - x_1^{(3)} & x_2^{(1)} - x_2^{(3)} \\ x_1^{(2)} - x_1^{(3)} & x_2^{(2)} - x_2^{(3)} \end{bmatrix} \qquad (180)$$

- Thus the determinant becomes twice the area of triangle element.

$$\begin{aligned} \det[J] &= \frac{\partial x_1}{\partial r_1}\frac{\partial x_2}{\partial r_2} - \frac{\partial x_2}{\partial r_1}\frac{\partial x_1}{\partial r_2} \\ &= (x_1^{(1)} - x_1^{(3)})(x_2^{(2)} - x_2^{(3)}) - (x_2^{(1)} - x_2^{(3)})(x_1^{(2)} - x_1^{(3)}) \end{aligned} \qquad (181)$$

$$\int_0^1 \int_0^{1-r_1} [B]^T [D] [B] \det[J] dr_2 dr_1$$
$$= \det[J] \int_0^1 \int_0^{1-r_1} [B]^T [D] [B] \, dr_2 dr_1 \qquad (182)$$

Procedure of Drawing Stiffness Matrix– Triangle 5

- $[B]$ Matrix components are consisted of the series of $\dfrac{\partial N^{(n)}}{\partial x_i}$, and evaluated by following,

$$\begin{bmatrix} \dfrac{\partial N^{(n)}}{\partial x_1} \\ \dfrac{\partial N^{(n)}}{\partial x_2} \end{bmatrix} = [J]^{-1} \begin{bmatrix} \dfrac{\partial N^{(n)}}{\partial r_1} \\ \dfrac{\partial N^{(n)}}{\partial r_2} \end{bmatrix}$$

$$= \frac{1}{\det[J]} \begin{bmatrix} x_2^{(2)} - x_2^{(3)} & -(x_2^{(1)} - x_2^{(3)}) \\ -(x_1^{(2)} - x_1^{(3)}) & x_1^{(1)} - x_1^{(3)} \end{bmatrix} \begin{bmatrix} \dfrac{\partial N^{(n)}}{\partial r_1} \\ \dfrac{\partial N^{(n)}}{\partial r_2} \end{bmatrix} \quad (183)$$

Since $\frac{\partial N^{(n)}}{\partial r_i}$ are all invariables, then automatically $\frac{\partial N^{(n)}}{\partial x_i}$ can be considered as invariables as well,

$$\begin{aligned}
&\begin{bmatrix} \frac{\partial N^{(1)}}{\partial x_1} & \frac{\partial N^{(2)}}{\partial x_1} & \frac{\partial N^{(3)}}{\partial x_1} \\ \frac{\partial N^{(1)}}{\partial x_2} & \frac{\partial N^{(2)}}{\partial x_2} & \frac{\partial N^{(3)}}{\partial x_2} \end{bmatrix} \\
&= \frac{1}{\det[J]} \begin{bmatrix} x_2^{(2)} - x_2^{(3)} & -(x_2^{(1)} - x_2^{(3)}) \\ -(x_1^{(2)} - x_1^{(3)}) & x_1^{(1)} - x_1^{(3)} \end{bmatrix} \begin{bmatrix} \frac{\partial N^{(1)}}{\partial r_1} & \frac{\partial N^{(2)}}{\partial r_1} & \frac{\partial N^{(3)}}{\partial r_1} \\ \frac{\partial N^{(1)}}{\partial r_2} & \frac{\partial N^{(2)}}{\partial r_2} & \frac{\partial N^{(3)}}{\partial r_2} \end{bmatrix} \\
&= \frac{1}{\det[J]} \begin{bmatrix} x_2^{(2)} - x_2^{(3)} & -(x_2^{(1)} - x_2^{(3)}) \\ -(x_1^{(2)} - x_1^{(3)}) & x_1^{(1)} - x_1^{(3)} \end{bmatrix} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \end{bmatrix} \\
&= \frac{1}{\det[J]} \begin{bmatrix} x_2^{(2)} - x_2^{(3)} & x_2^{(3)} - x_2^{(1)} & x_2^{(1)} - x_2^{(2)} \\ x_1^{(3)} - x_1^{(2)} & x_1^{(1)} - x_1^{(3)} & x_1^{(2)} - x_1^{(1)} \end{bmatrix}
\end{aligned} \tag{184}$$

Procedure of Drawing Stiffness Matrix– Triangle 6

- Therefore,$[B]$ matrix components are specifically described as,

$$[B]=\begin{bmatrix}\frac{\partial N^{(1)}}{\partial x_1} & 0 & \frac{\partial N^{(2)}}{\partial x_1} & 0 & \frac{\partial N^{(3)}}{\partial x_1} & 0\\ 0 & \frac{\partial N^{(1)}}{\partial x_2} & 0 & \frac{\partial N^{(2)}}{\partial x_2} & 0 & \frac{\partial N^{(3)}}{\partial x_2}\\ \frac{\partial N^{(1)}}{\partial x_2} & \frac{\partial N^{(1)}}{\partial x_1} & \frac{\partial N^{(2)}}{\partial x_2} & \frac{\partial N^{(2)}}{\partial x_1} & \frac{\partial N^{(3)}}{\partial x_2} & \frac{\partial N^{(3)}}{\partial x_1}\end{bmatrix}$$
$$=\frac{1}{\det[J]}\begin{bmatrix}x_2^{(2)}-x_2^{(3)} & 0 & x_2^{(3)}-x_2^{(1)} & 0 & x_2^{(1)}-x_2^{(2)} & 0\\ 0 & x_1^{(3)}-x_1^{(2)} & 0 & x_1^{(1)}-x_1^{(3)} & 0 & x_1^{(2)}-x_1^{(1)}\\ x_1^{(3)}-x_1^{(2)} & x_2^{(2)}-x_2^{(3)} & x_1^{(1)}-x_1^{(3)} & x_2^{(3)}-x_2^{(1)} & x_1^{(2)}-x_1^{(1)} & x_2^{(1)}-x_2^{(2)}\end{bmatrix} \tag{185}$$

- Moreover, in realizing $[D]$ matrix of being invariables, then automatically integrand all becomes invariables as well. There is no more need for the numerical integrations.

$$\begin{aligned}&\int_0^1\int_0^{1-r_1}[B]^T\,[D]\,[B]\det[J]dr_2dr_1\\ =&\det[J]\int_0^1\int_0^{1-r_1}[B]^T\,[D]\,[B]\,dr_2dr_1\\ =&\det[J][B]^T\,[D]\,[B]\int_0^1\int_0^{1-r_1}dr_2dr_1\\ =&\frac{1}{2}\det[J][B]^T\,[D]\,[B]\end{aligned} \tag{186}$$

External Force Vector

- In actual finite element analysis, the external force vectors are needed to be obtained.

$$\begin{aligned}\int_{\partial\Omega_e} \delta u_i \cdot t_i \,\mathrm{d}S &= \int_{\partial\Omega_e} \{\delta u_i^{(n)}\}^T [N]^T \{t\} \,\mathrm{d}S \\ &= \{\delta u_i^{(n)}\}^T \int_{\partial\Omega_e} [N]^T \{t\} \,\mathrm{d}S \end{aligned} \quad (187)$$

$$\begin{aligned}\int_{\Omega_e} \rho\, \delta u_i \cdot g_i \,\mathrm{d}\Omega &= \int_{\Omega_e} \rho \{\delta u_i^{(n)}\}^T [N]^T \{g\} \,\mathrm{d}\Omega \\ &= \{\delta u_i^{(n)}\}^T \int_{\Omega_e} \rho [N]^T \{g\} \,\mathrm{d}\Omega \end{aligned} \quad (188)$$

- Provided that,

$$[N] = \begin{bmatrix} N^{(1)} & & & N^{(2)} & & & & N^{(n)} & & \\ & N^{(1)} & & & N^{(2)} & & \dots & & N^{(n)} & \\ & & N^{(1)} & & & N^{(2)} & & & & N^{(n)} \end{bmatrix} \quad (189)$$

$$\{t\} = \begin{Bmatrix} t_1 \\ t_2 \\ t_3 \end{Bmatrix}, \quad \{g\} = \begin{Bmatrix} g_1 \\ g_2 \\ g_3 \end{Bmatrix} \quad (190)$$

- Based on the fact, the external force vector $\{F^{(e)}\}$ can be defined as,

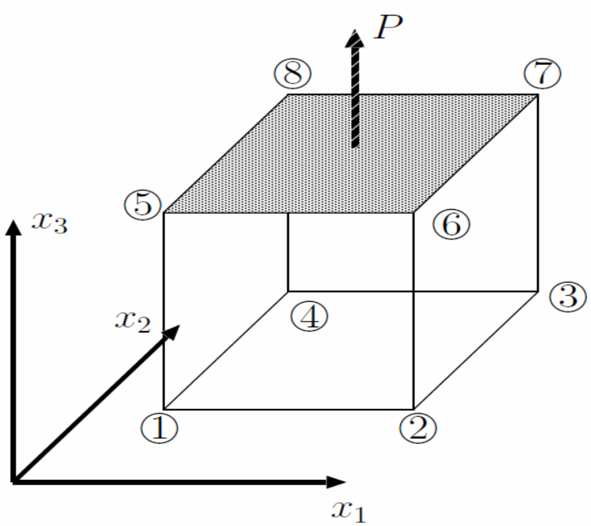
$$\{F^{(e)}\} = \int_{\partial\Omega_e} [N]^T \{t\} \,\mathrm{d}S + \int_{\Omega_e} \rho [N]^T \{g\} \,\mathrm{d}\Omega \quad (191)$$

Example of External Force 1

- Example of external force, pull by the equal force $-x_3$ direction

$$\{F^{(e)}\} = \int_{\partial\Omega_e} [N] \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} dS \qquad (192)$$

- In order to simplify, we suppose the weight loaded on the plane 5 -6 -7 -8 parallel to the plane $x1 - x2$ then set the coordinates for each nodal points.



⑤-⑥//⑦-⑧//x_1 軸, ⑥-⑦//⑧-⑤//x_2 軸,

$$x_1^{(6)} = x_1^{(5)} + a \qquad (193)$$

$$x_1^{(7)} = x_1^{(5)} + a \qquad (194)$$

$$x_1^{(8)} = x_1^{(5)} \qquad (195)$$

$$x_2^{(6)} = x_2^{(5)} \qquad (196)$$

$$x_2^{(7)} = x_2^{(5)} + b \qquad (197)$$

$$x_2^{(8)} = x_2^{(5)} + b \qquad (198)$$

図 7: 単純引っ張り

Hence, it is considered as rectangular by $a \times b$. Having $r3 = 1$ on the plane 5 -6 -7 -8, becomes as,

$$\frac{\partial N^{(5)}}{\partial r_1} = -\frac{1}{4}(1 - r_2) \tag{199}$$

$$\frac{\partial N^{(5)}}{\partial r_2} = -\frac{1}{4}(1 - r_1) \tag{200}$$

$$\frac{\partial N^{(6)}}{\partial r_1} = \frac{1}{4}(1 - r_2) \tag{201}$$

$$\frac{\partial N^{(6)}}{\partial r_2} = -\frac{1}{4}(1 + r_1) \tag{202}$$

$$\frac{\partial N^{(7)}}{\partial r_1} = \frac{1}{4}(1 + r_2) \tag{203}$$

$$\frac{\partial N^{(7)}}{\partial r_2} = \frac{1}{4}(1 + r_1) \tag{204}$$

$$\frac{\partial N^{(8)}}{\partial r_1} = -\frac{1}{4}(1 + r_2) \tag{205}$$

$$\frac{\partial N^{(8)}}{\partial r_2} = \frac{1}{4}(1 - r_1) \tag{206}$$

Example of External Force 2

- Based on the fact $\frac{\partial x}{\partial r}$ can be gained by following the steps below.

$$\begin{aligned}
\frac{\partial x_1}{\partial r_1} &= -\frac{1}{4}(1-r_2)x_1^{(5)} + \frac{1}{4}(1-r_2)x_1^{(6)} + \frac{1}{4}(1+r_2)x_1^{(7)} - \frac{1}{4}(1+r_2)x_1^{(8)} \\
&= -\frac{1}{4}(1-r_2)x_1^{(5)} + \frac{1}{4}(1-r_2)(x_1^{(5)}+a) + \frac{1}{4}(1+r_2)(x_1^{(5)}+a) - \frac{1}{4}(1+r_2)x_1^{(5)} \\
&= \frac{a}{2} \qquad (207)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x_2}{\partial r_1} &= -\frac{1}{4}(1-r_2)x_2^{(5)} + \frac{1}{4}(1-r_2)x_2^{(6)} + \frac{1}{4}(1+r_2)x_2^{(7)} - \frac{1}{4}(1+r_2)x_2^{(8)} \\
&= -\frac{1}{4}(1-r_2)x_2^{(5)} + \frac{1}{4}(1-r_2)x_2^{(5)} + \frac{1}{4}(1+r_2)(x_2^{(5)}+b) - \frac{1}{4}(1+r_2)(x_2^{(5)}+b) \\
&= 0 \qquad (208)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x_1}{\partial r_2} &= -\frac{1}{4}(1-r_1)x_1^{(5)} - \frac{1}{4}(1+r_1)x_1^{(6)} + \frac{1}{4}(1+r_1)x_1^{(7)} + \frac{1}{4}(1-r_1)x_1^{(8)} \\
&= -\frac{1}{4}(1-r_1)x_1^{(5)} - \frac{1}{4}(1+r_1)(x_1^{(5)}+a) + \frac{1}{4}(1+r_1)(x_1^{(5)}+a) + \frac{1}{4}(1-r_1)x_1^{(5)} \\
&= 0 \qquad (209)
\end{aligned}$$

$$\begin{aligned}
\frac{\partial x_2}{\partial r_2} &= -\frac{1}{4}(1-r_1)x_2^{(5)} - \frac{1}{4}(1+r_1)x_2^{(6)} + \frac{1}{4}(1+r_1)x_2^{(7)} + \frac{1}{4}(1-r_1)x_2^{(8)} \\
&= -\frac{1}{4}(1-r_1)x_2^{(5)} - \frac{1}{4}(1+r_1)x_2^{(5)} + \frac{1}{4}(1+r_1)(x_2^{(5)}+b) + \frac{1}{4}(1-r_1)(x_2^{(5)}+b) \\
&= \frac{b}{2} \qquad (210)
\end{aligned}$$

Example of External Force 3

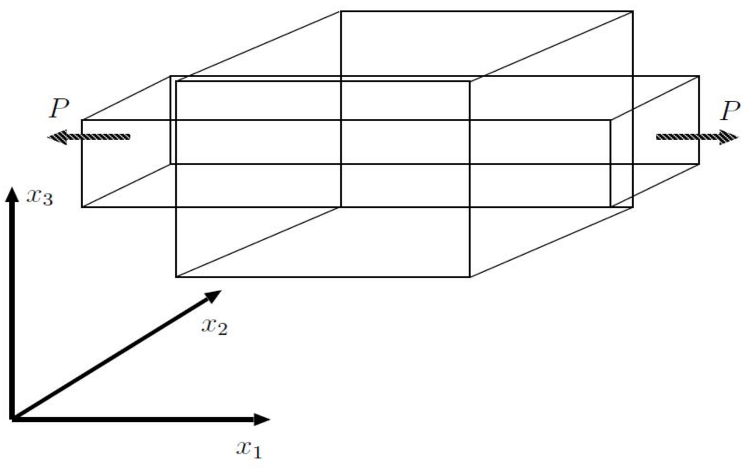
- Thus if we define the load per unit area as P, we can find the external force at each nodal points

$$\begin{aligned}
\{F^{(e)}\} &= \int_{\partial\Omega_e} [N] \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} dS && (211) \\
&= \int_{-1}^{1}\int_{-1}^{1} \begin{bmatrix} N^{(1)} & & & \cdots & N^{(8)} & & \\ & N^{(1)} & & \cdots & & N^{(8)} & \\ & & N^{(1)} & \cdots & & & N^{(8)} \end{bmatrix}^T \begin{Bmatrix} 0 \\ 0 \\ P \end{Bmatrix} \frac{ab}{4}\, dr_1\, dr_2 && (212) \\
F_1^{(i)} &= F_2^{(i)} = 0 \quad (i = 1 \sim 8) && (213) \\
F_3^{(i)} &= 0 \quad (i = 1 \sim 4) \quad (\because r_3 = 1) && (214) \\
F_3^{(5)} &= P \int_{-1}^{1}\int_{-1}^{1} N^{(5)} \frac{ab}{4}\, dr_1\, dr_2 && (215) \\
&= P \int_{-1}^{1}\int_{-1}^{1} \frac{1}{4}(1-r_1)(1-r_2)\frac{ab}{4}\, dr_1\, dr_2 = \frac{abP}{4} && (216) \\
F_3^{(6)} &= P \int_{-1}^{1}\int_{-1}^{1} N^{(6)} \frac{ab}{4}\, dr_1\, dr_2 && (217) \\
&= P \int_{-1}^{1}\int_{-1}^{1} \frac{1}{4}(1+r_1)(1-r_2)\frac{ab}{4}\, dr_1\, dr_2 = \frac{abP}{4} && (218) \\
F_3^{(7)} &= P \int_{-1}^{1}\int_{-1}^{1} N^{(7)} \frac{ab}{4}\, dr_1\, dr_2 && (219) \\
&= P \int_{-1}^{1}\int_{-1}^{1} \frac{1}{4}(1+r_1)(1+r_2)\frac{ab}{4}\, dr_1\, dr_2 = \frac{abP}{4} && (220) \\
F_3^{(8)} &= P \int_{-1}^{1}\int_{-1}^{1} N^{(8)} \frac{ab}{4}\, dr_1\, dr_2 && (221) \\
&= P \int_{-1}^{1}\int_{-1}^{1} \frac{1}{4}(1-r_1)(1+r_2)\frac{ab}{4}\, dr_1\, dr_2 = \frac{abP}{4} && (222)
\end{aligned}$$

- In summary, considering the case where uniformly distributed load is acted on the plane of 8 noded hexahedron solid element, we can assign ¼ of the load to each node point of total 4 nodes.

Boundary Condition 1

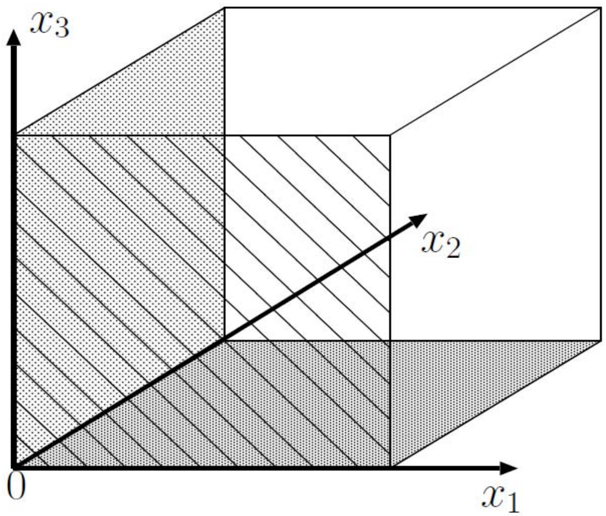
- In conducting stationary structural analysis in finite element method, the solving equation may be the balance equation, thus we need to set an arbitrary fix point to solve the equation.
- For a simple pull, the two planes of the rectangular solid counterparts are simply pulled, and we cannot determine the position of a center point by any means.



- Let us now consider the formulation, we can realize there are three symmetrical planes in this formulation if we assume the central mass points to stay where it is.

Boundary Condition 1

- Suppose the coordinates of the central mass point in rectangular solid to be (0,0,0), then take out the the first quadrant as one of the 1/8 region divided by the three symmetrical planes, we can obtain the following boundary condition.



$x_1 - x_2$ 平面上の点:
x_3 方向固定
$x_2 - x_3$ 平面上の点:
x_1 方向固定
$x_3 - x_1$ 平面上の点:
x_2 方向固定

図 8: 1/8 対象モデル

Merge

- For the bugs occur in a merge operation by adding an element stiffness to a total stiffness can be considered as a mistake of an index.
- Debugging of merge is carried by plugging some simple values as 1, and 2 into the element stiffness then shift over to the total stiffness to verify the result.
- In order to have a variable degree of freedom per single node point, we may implement following coding to the merge.

$$\textbf{for} \quad i = 1 \sim \texttt{nelem} \tag{223}$$

$$\quad \text{call element}(i) \tag{224}$$

$$\quad \textbf{for} \quad j = 1 \sim \texttt{ntnoel}(i) \tag{225}$$

$$\quad\quad \textbf{for} \quad k = 1 \sim \texttt{ndofn} \tag{226}$$

$$\quad\quad\quad \text{ip}((j-1)\cdot\texttt{ndofn}+k) = (\texttt{lnods}(j,i)-1)\cdot\texttt{ndofn}+k \tag{227}$$

$$\quad\quad \textbf{end for} \tag{228}$$

$$\quad \textbf{end for} \tag{229}$$

$$\quad \textbf{for} \quad j = 1 \sim \texttt{ntnoel}(i)\cdot\texttt{ndofn} \tag{230}$$

$$\quad\quad \textbf{for} \quad k = 1 \sim \texttt{ntnoel}(i)\cdot\texttt{ndofn} \tag{231}$$

$$\quad\quad\quad K_{ip(j),ip(k)} = K_{ip(j),ip(k)} + A_{jk} \tag{232}$$

$$\quad\quad \textbf{end for} \tag{233}$$

$$\quad \textbf{end for} \tag{234}$$

$$\textbf{end for} \tag{235}$$

Gauss Elimination

- The coefficient matrix $[A]$ given by Gauss elimination may finally be decomposed in to the following equations,

$$[A] = [L][D][U] \tag{236}$$

- If $[A]$ represents a symmetric matrix, then we get $[U] = [LT]$. Therefore, $[A]$can be decomposed in the following equations.

$$[A] = [L][D][L^T] = [U^T][D][U] \tag{237}$$

- By obtaining $[A] = [L][D][U]$, the system of linear equation $[A]\{b\} = \{c\}$ is solved by adopting inverse matrix on both sides in turn.

$$\begin{aligned}
[L][D][U]\{b\} &= \{c\} \\
[D][U]\{b\} &= [L^{-1}]\{c\} \\
[U]\{b\} &= [D^{-1}][L^{-1}]\{c\} \\
\{b\} &= [U^{-1}][D^{-1}][L^{-1}]\{c\}
\end{aligned} \tag{238}$$

Boundary Condition Handling 4

- Consider now for the following system of linear equations,

$$\begin{bmatrix} A_{11} & A_{12} & \dots & A_{1n} \\ A_{21} & & & \vdots \\ \vdots & & & \vdots \\ A_{n1} & \dots & \dots & A_{nn} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_n \end{Bmatrix} \tag{239}$$

- Here, we assume the right hand side other than *Cj* and *bj* are the known. Re-write in the system of linear equations formulation, it is obtained by ,

$$\begin{cases} A_{11} + A_{12}b_2 + \cdots + A_{1j}b_j + \cdots + A_{1n}b_n & = C_1 \\ A_{21} + A_{22}b_2 + \cdots + A_{2j}b_j + \cdots + A_{2n}b_n & = C_2 \\ \vdots & \vdots \\ A_{j1} + A_{j2}b_2 + \cdots + A_{jj}b_j + \cdots + A_{jn}b_n & = C_j \\ \vdots & \vdots \\ A_{n1} + A_{n2}b_2 + \cdots + A_{nj}b_j + \cdots + A_{nn}b_n & = C_n \end{cases} \tag{240}$$

- In above, the equation in the column *j* contains unknown quantity *Cj* in the right side, thus *Cj* is obtained after the ordinary unknown quantity *bi*(*i* = 1 ~ *j* –1*, j*+1 ~ *n*) are obtained based on the following.

$$C_j = A_{j1}b_1 + A_{j2}b_2 + \cdots + Ajjb_j + \cdots + A_{jn}b_n \tag{241}$$

For its distinct in nature compared to other equations in the column, $b_i(i = 1 \sim j-1, j+1 \sim n)$ should be excluded from the process.

Boundary Condition Handling 5

- Consider the $n-1$ equations without column j .

$$\left\{\begin{array}{ccccccccccc}
A_{11} & + & A_{12}b_2 & + & \cdots & + & A_{1j}b_j & + & \cdots & + & A_{1n}b_n & = & C_1 \\
A_{21} & + & A_{22}b_2 & + & \cdots & + & A_{2j}b_j & + & \cdots & + & A_{2n}b_n & = & C_2 \\
 & & & & & & \vdots & & & & & & \vdots \\
A_{j-1\,1} & + & A_{j-1\,2}b_2 & + & \cdots & + & A_{j-1\,j}b_j & + & \cdots & + & A_{j-1\,n}b_n & = & C_{j-1} \\
A_{j+1\,1} & + & A_{j+1\,2}b_2 & + & \cdots & + & A_{j+1\,j}b_j & + & \cdots & + & A_{j-1\,n}b_n & = & C_{j+1} \\
 & & & & & & \vdots & & & & & & \vdots \\
A_{n1} & + & A_{n2}b_2 & + & \cdots & + & A_{nj}b_j & + & \cdots & + & A_{nn}b_n & = & C_n
\end{array}\right. \qquad (242)$$

- In the equation above, $A_{1j}b_j, A_{2j}b_j, \cdots, A_{nj}b_j$ are known value and should be transpositioned into the right hand side.

$$\left\{\begin{array}{ccccccc}
A_{11}+A_{12}b_2 & +\cdots+ & A_{1\,j-1}b_{j-1}+A_{1\,j+1}b_{j+1} & +\cdots+ & A_{1n}b_n & = & C_1-A_{1j}b_j \\
A_{21}+A_{22}b_2 & +\cdots+ & A_{2\,j-1}b_{j-1}+A_{2\,j+1}b_{j+1} & +\cdots+ & A_{2n}b_n & = & C_2-A_{2j}b_j \\
 & & \vdots & & & & \vdots \\
A_{j-1\,1}+A_{j-1\,2}b_2 & +\cdots+ & A_{j-1\,j-1}b_{j-1}+A_{j-1\,j+1}b_{j+1} & +\cdots+ & A_{j-1\,n}b_n & = & C_{j-1}-A_{j-1\,j}b_j \\
A_{j+1\,1}+A_{j+1\,2}b_2 & +\cdots+ & A_{j+1\,j-1}b_{j-1}+A_{j+1\,j+1}b_{j+1} & +\cdots+ & A_{j+1\,n}b_n & = & C_{j+1}-A_{j+1\,j}b_j \\
 & & \vdots & & & & \vdots \\
A_{n1}+A_{n2}b_2 & +\cdots+ & A_{n\,j-1}b_{j-1}+A_{n\,j+1}b_{j+1} & +\cdots+ & A_{nn}b_n & = & C_n-A_{nj}b_j
\end{array}\right. \qquad (243)$$

Boundary Condition Handling 6

- To formally express in matrix, we have the following.
- Coefficient matrix consists of the matrix without column *j* and the row *j* .
- Unknown vector represents the one without column *j.*
- The right hand side is consisted of the initial left hand side subtracted by *bj* times the initial coefficient matrix row *j* without the column *j*.

$$\begin{bmatrix} A_{11} & \dots & A_{1\,j-1} & A_{1\,j+1} & \dots & A_{1n} \\ A_{21} & \dots & A_{2\,j-1} & A_{2\,j+1} & \dots & A_{2n} \\ \vdots & & & & & \\ A_{j-1\,1} & \dots & A_{j-1\,j-1} & A_{j+1\,j+1} & \dots & A_{j-1\,n} \\ A_{j+1\,1} & \dots & A_{j+1\,j+1} & A_{j+1\,j+1} & \dots & A_{j+1\,n} \\ \vdots & & & & & \\ A_{n1} & \dots & A_{n\,j-1} & A_{n\,j+1} & \dots & A_{nn} \end{bmatrix} \begin{Bmatrix} b_1 \\ b_2 \\ \vdots \\ b_{j-1} \\ b_{j+1} \\ \vdots \\ b_n \end{Bmatrix} = \begin{Bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{j-1} \\ C_{j+1} \\ \vdots \\ C_n \end{Bmatrix} - b_j \begin{Bmatrix} A_{1j} \\ A_{2j} \\ \vdots \\ A_{j-1\,j} \\ A_{j+1\,j} \\ \vdots \\ A_{nj} \end{Bmatrix} \qquad (244)$$

Where b_j is 0, the right hand side the second term can be eliminated.

- The operation above stays the same with multiple known b_j *.* Suppose $b_{j(1)},\ b_{j(2)}\ ,\cdots,\ b_{j(m)}$ are know. From the system of linear equations, exclude the equations in column $j(1),\ j(2),\ \cdots,\ j(m)$. Then transposition the terms, which include $b_{j(1)}, b_{j(2)}, \cdots, b_{j(m)}$ in the left side to the right hand side. Based on the fact, the matrix can be obtained.