Nonlinear Finite Element Method

13/12/2004

Nonlinear Finite Element Method

- Lectures include discussion of the nonlinear finite element method.
- It is preferable to have completed “Introduction to Nonlinear Finite Element Analysis” available in summer session.
- If not, students are required to study on their own before participating this course. Reference:Toshiaki.,Kubo. “Introduction: Tensor Analysis For Nonlinear Finite Element Method” (Hisennkei Yugen Yoso no tameno Tensor Kaiseki no Kiso),Maruzen.
- Lecture references are available and downloadable at http://www.sml.k.u-tokyo.ac.jp/members/nabe/lecture2004 They should be posted on the website by the day before scheduled meeting, and each students are expected to come in with a copy of the reference.
- Lecture notes from previous year are available and downloadable, also at http://www.sml.k.u.tokyo.ac.jp/members/nabe/lecture2003 You may find the course title, ”Advanced Finite Element Method” but the contents covered are the same I will cover this year.
- I will assign the exercises from this year, and expect the students to hand them in during the following lecture. They are not the requirements and they will not be graded, however it is important to actually practice calculate in deeper understanding the finite element method.
- For any questions, contact me at nabe@sml.k.u-tokyo.ac.jp

Nonlinear Finite Element Method Lecture Schedule

1. 10/ 4 Finite element analysis in boundary value problems and the differential equations
2. 10/18 Finite element analysis in linear elastic body
3. 10/25 Isoparametric solid element (program)
4. 11/ 1 Numerical solution and boundary condition processing for system of linear equations（with exercises）
5. 11/ 8 Basic program structure of the linear finite element method(program)
6. 11/15 Finite element formulation in geometric nonlinear problems(program)
7. 11/22 Static analysis technique、hyperelastic body and elastic-plastic material for nonlinear equations (program)
8. 11/29 Exercises for Lecture7
9. 12/ 6 Dynamic analysis technique and eigenvalue analysis in the nonlinear equations
10. 12/13 Structural element
11. 12/20 Numerical solution— skyline method、iterative method for the system of linear equations
12. 1/17 ALE finite element fluid analysis
13. 1/24 ALE finite element fluid analysis

Beam Element in Cross Section of Rectangle

- In a simple discritization of the common partial derivative equations,a solid element is thought to be more natural.
- However, in a structure analysis, there are other views to be considered.
- Beam element is considered as a typical structure element.
- Which is adopted in analysis of a structure consisted of the components that appear to be slender and smaller than the measurement of the stretch of cross section.
- In terms of modeling the structure, adopting the beam element may facilitate the process of modeling in comparison with when adopting the solid element.
- Locking may occur when low order solid element is applied to model the beam.
- At time 0, a plane perpendicular to the neutral axis of the beam may hold on to the plane while deformation is being carried out, however, the plane does not necessarily have to be perpendicular to a deformed neutral axis.
- There is no deformation in cross section of beam.

Interpolation Function for Coordinates and Displacement 1

- With time t and nodal point n are given, vectors in cross section are defined as ${}^t\boldsymbol{V}_2^{(n)}$, ${}^t\boldsymbol{V}_3^{(n)}$, . In addition to the position vector given as ${}^t\boldsymbol{x}$ at time t for the nodal point n.

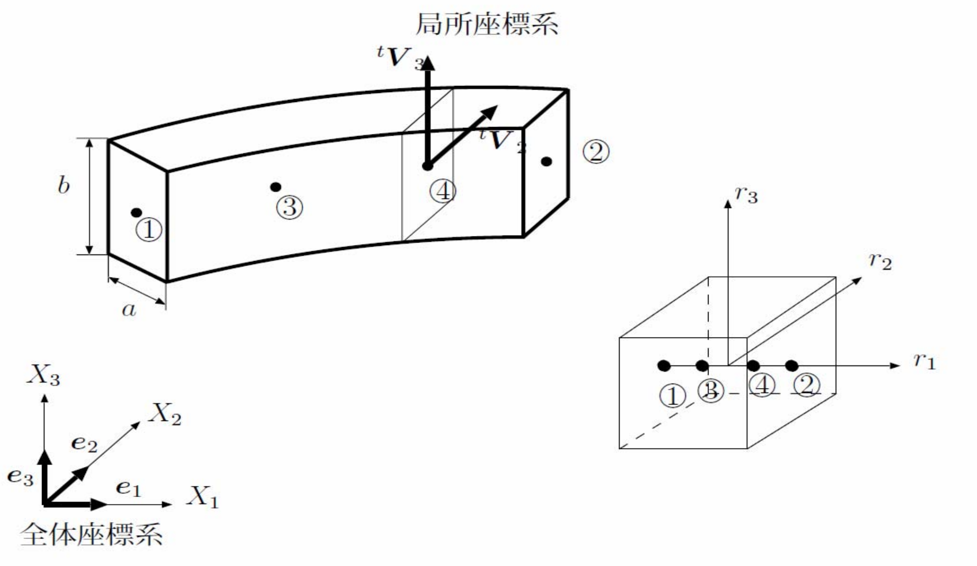

図 1: 座標系の定義

- The position vector for the arbitrary points within the element at time t can be expressed as,

$$
{}^t\boldsymbol{x}(r_1, r_2, r_3) = N^{(n)}(r_1)\left({}^t\boldsymbol{x}^n + \frac{a}{2}r_2{}^t\boldsymbol{V}_2^{(n)} + \frac{b}{2}r_3{}^t\boldsymbol{V}_3^{(n)}\right) \quad (1)
$$

- Where single dimensional element is used for shape function $N(n)$.

Interpolation Function For Coordinates and Displacement 2

- The local coordinate vectors ${}^{t}\boldsymbol{V}_2^{(n)}, {}^{t}\boldsymbol{V}_3^{(n)}$ are fixed in the cross section of beam, and considering the fact the vectors may rotate along with the deformation of element, at time t form $t'(= t + \Delta t)$, the deformation increment vector $\boldsymbol{u}$ may be given by,

$$\begin{aligned}\boldsymbol{u} &= {}^{t'}\boldsymbol{x} - {}^{t}\boldsymbol{x} \\ &= N^{(n)}\left\{{}^{t'}\boldsymbol{x}^{(n)} - {}^{t}\boldsymbol{x}^{(n)} + \frac{a}{2}r_2\left({}^{t'}\boldsymbol{V}_2^{(n)} - {}^{t}\boldsymbol{V}_2^{(n)}\right) + \frac{b}{2}r_3\left({}^{t'}\boldsymbol{V}_3^{(n)} - {}^{t}\boldsymbol{V}_3^{(n)}\right)\right\}\end{aligned}$$

- Here, suppose the rotation increment is infinitesimal at Δt, then expressed by,

$$ {}^{t'}\boldsymbol{V}_i^{(n)} \simeq {}^{t}\boldsymbol{V}_i^{(n)} + \theta^{(n)} \times {}^{t}\boldsymbol{V}_i^{(n)} \tag{2}$$

- Therefore,

$$\begin{aligned}{}^{t'}\boldsymbol{V}_i^{(n)} - {}^{t}\boldsymbol{V}_i^{(n)} &= \theta^{(n)} \times {}^{t}\boldsymbol{V}_i^{(n)} \\ &= \begin{bmatrix} 0 & -\theta_3^{(n)} & \theta_2^{(n)} \\ \theta_3^{(n)} & 0 & -\theta_1^{(n)} \\ -\theta_2^{(n)} & \theta_1^{(n)} & 0 \end{bmatrix} \begin{Bmatrix} {}^{t}V_{i1}^{(n)} \\ {}^{t}V_{i2}^{(n)} \\ {}^{t}V_{i3}^{(n)} \end{Bmatrix} = \begin{Bmatrix} \theta_2^{(n)t}V_{i3}^{(n)} - \theta_3^{(n)t}V_{i2}^{(n)} \\ \theta_3^{(n)t}V_{i1}^{(n)} - \theta_1^{(n)t}V_{i3}^{(n)} \\ \theta_1^{(n)t}V_{i2}^{(n)} - \theta_2^{(n)t}V_{i1}^{(n)} \end{Bmatrix} \\ &= \begin{bmatrix} 0 & {}^{t}V_{i3}^{(n)} & -{}^{t}V_{i2}^{(n)} \\ -{}^{t}V_{i3}^{(n)} & 0 & {}^{t}V_{i1}^{(n)} \\ {}^{t}V_{i2}^{(n)} & -{}^{t}V_{i1}^{(n)} & 0 \end{bmatrix} \begin{Bmatrix} \theta_1^{(n)} \\ \theta_2^{(n)} \\ \theta_3^{(n)} \end{Bmatrix}\end{aligned} \tag{3}$$

Interpolation Function For Coordinates and Displacement 3

- Now, let us write out the components of ${}^t\boldsymbol{x}, \boldsymbol{u}$ at total coordinates system,

$$\begin{Bmatrix} {}^tx_1 \\ {}^tx_2 \\ {}^tx_3 \end{Bmatrix} = N^{(n)}(r_1) \begin{Bmatrix} {}^tx_1^{(n)} \\ {}^tx_2^{(n)} \\ {}^tx_3^{(n)} \end{Bmatrix} + \frac{a}{2} r_2 N^{(n)}(r_1) \begin{Bmatrix} {}^tV_{21}^{(n)} \\ {}^tV_{22}^{(n)} \\ {}^tV_{23}^{(n)} \end{Bmatrix} + \frac{b}{2} r_3 N^{(n)}(r_1) \begin{Bmatrix} {}^tV_{31}^{(n)} \\ {}^tV_{32}^{(n)} \\ {}^tV_{33}^{(n)} \end{Bmatrix} \tag{4}$$

$$\begin{aligned} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = N^{(n)}(r_1) \begin{Bmatrix} u_1^{(n)} \\ u_2^{(n)} \\ u_3^{(n)} \end{Bmatrix} &+ \frac{a}{2} r_2 N^{(n)}(r_1) \begin{bmatrix} 0 & {}^tV_{23}^{(n)} & -{}^tV_{22}^{(n)} \\ -{}^tV_{23}^{(n)} & 0 & {}^tV_{21}^{(n)} \\ {}^tV_{22}^{(n)} & -{}^tV_{21}^{(n)} & 0 \end{bmatrix} \begin{Bmatrix} \theta_1^{(n)} \\ \theta_2^{(n)} \\ \theta_3^{(n)} \end{Bmatrix} \\ &+ \frac{b}{2} r_3 N^{(n)}(r_1) \begin{bmatrix} 0 & {}^tV_{33}^{(n)} & -{}^tV_{32}^{(n)} \\ -{}^tV_{33}^{(n)} & 0 & {}^tV_{31}^{(n)} \\ {}^tV_{32}^{(n)} & -{}^tV_{31}^{(n)} & 0 \end{bmatrix} \begin{Bmatrix} \theta_1^{(n)} \\ \theta_2^{(n)} \\ \theta_3^{(n)} \end{Bmatrix} \end{aligned} \tag{5}$$

Strain Calculation 1

- The quantity needed in evaluating the strain is $\partial u_i/\partial x_j$, yet the quantities that may be directly calculated are $\partial u_i/\partial r_j$, , $\partial x_i/\partial r_j$ so, adopt $\partial u_i/\partial r_j$ to transform $\partial x_i/\partial r_j$.
- The relation between the unknown quantity $\partial u_i/\partial x_j$ and the known quantity $\partial u_i/\partial r_j$ can be expressed by,

$$\begin{Bmatrix} \frac{\partial u_i}{\partial r_1} \\ \frac{\partial u_i}{\partial r_2} \\ \frac{\partial u_i}{\partial r_3} \end{Bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_2}{\partial r_1} & \frac{\partial x_3}{\partial r_1} \\ \frac{\partial x_1}{\partial r_2} & \frac{\partial x_2}{\partial r_2} & \frac{\partial x_3}{\partial r_2} \\ \frac{\partial x_1}{\partial r_3} & \frac{\partial x_2}{\partial r_3} & \frac{\partial x_3}{\partial r_3} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_i}{\partial x_1} \\ \frac{\partial u_i}{\partial x_2} \\ \frac{\partial u_i}{\partial x_3} \end{Bmatrix} \quad (i = 1, 2, 3) \tag{6}$$

- Here, given the Jacobian matrix $[J]$,

$$\begin{bmatrix} J \end{bmatrix} = \begin{bmatrix} J_{11} & J_{12} & J_{13} \\ J_{21} & J_{22} & J_{23} \\ J_{31} & J_{32} & J_{33} \end{bmatrix} = \begin{bmatrix} \frac{\partial x_1}{\partial r_1} & \frac{\partial x_2}{\partial r_1} & \frac{\partial x_3}{\partial r_1} \\ \frac{\partial x_1}{\partial r_2} & \frac{\partial x_2}{\partial r_2} & \frac{\partial x_3}{\partial r_2} \\ \frac{\partial x_1}{\partial r_3} & \frac{\partial x_2}{\partial r_3} & \frac{\partial x_3}{\partial r_3} \end{bmatrix} \tag{7}$$

- By multiplying the inverse matrix of $[J]$ to the equation (6) of both sides from the felt,

$$\begin{Bmatrix} \frac{\partial u_i}{\partial x_1} \\ \frac{\partial u_i}{\partial x_2} \\ \frac{\partial u_i}{\partial x_3} \end{Bmatrix} = \begin{bmatrix} J_{11}^{-1} & J_{12}^{-1} & J_{13}^{-1} \\ J_{21}^{-1} & J_{22}^{-1} & J_{23}^{-1} \\ J_{31}^{-1} & J_{32}^{-1} & J_{33}^{-1} \end{bmatrix} \begin{Bmatrix} \frac{\partial u_i}{\partial r_1} \\ \frac{\partial u_i}{\partial r_2} \\ \frac{\partial u_i}{\partial r_3} \end{Bmatrix} \quad (i = 1, 2, 3) \tag{8}$$

- Jacobian components $\partial x_i / \partial r_i$ may be given by,

$$\frac{\partial^t x_i}{\partial r_1} = \frac{\partial N^{(n)}(r_1)}{\partial r_1} \left\{ {}^t x_i^{(n)} + \frac{a}{2} r_2 {}^t V_{2i}^{(n)} + \frac{b}{2} r_3 {}^t V_{3i}^{(n)} \right\} \tag{9}$$

$$\frac{\partial^t x_i}{\partial r_1} = \frac{a}{2} N^{(n)}(r_1) {}^t V_{2i}^{(n)} \tag{10}$$

$$\frac{\partial^t x_i}{\partial r_1} = \frac{b}{2} N^{(n)}(r_1) {}^t V_{3i}^{(n)} \tag{11}$$

Strain Calculation 2

- Moreover, derivative $\partial u_i/\partial r_j$ of deformation $\boldsymbol{u}$ can be as following.

$$\frac{\partial}{\partial r_1}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{\partial N^{(n)}(r_1)}{\partial r_1}\left[\begin{Bmatrix} u_1^{(n)} \\ u_2^{(n)} \\ u_3^{(n)} \end{Bmatrix} + \frac{a}{2}r_2\begin{bmatrix} 0 & V_{23}^{(n)} & -V_{22}^{(n)} \\ -V_{23}^{(n)} & 0 & V_{21}^{(n)} \\ V_{22}^{(n)} & -V_{21}^{(n)} & 0 \end{bmatrix}\begin{Bmatrix} \theta_1^{(n)} \\ \theta_2^{(n)} \\ \theta_3^{(n)} \end{Bmatrix} \right.$$
$$\left. + \frac{b}{2}r_3\begin{bmatrix} 0 & V_{33}^{(n)} & -V_{32}^{(n)} \\ -V_{33}^{(n)} & 0 & V_{31}^{(n)} \\ V_{32}^{(n)} & -V_{31}^{(n)} & 0 \end{bmatrix}\begin{Bmatrix} \theta_1^{(n)} \\ \theta_2^{(n)} \\ \theta_3^{(n)} \end{Bmatrix}\right] \quad (12)$$

$$\frac{\partial}{\partial r_2}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{a}{2}N^{(n)}(r_1)\begin{bmatrix} 0 & V_{23}^{(n)} & -V_{22}^{(n)} \\ -V_{23}^{(n)} & 0 & V_{21}^{(n)} \\ V_{22}^{(n)} & -V_{21}^{(n)} & 0 \end{bmatrix}\begin{Bmatrix} \theta_1^{(n)} \\ \theta_2^{(n)} \\ \theta_3^{(n)} \end{Bmatrix} \quad (13)$$

$$\frac{\partial}{\partial r_3}\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \frac{b}{2}N^{(n)}(r_1)\begin{bmatrix} 0 & V_{33}^{(n)} & -V_{32}^{(n)} \\ -V_{33}^{(n)} & 0 & V_{31}^{(n)} \\ V_{32}^{(n)} & -V_{31}^{(n)} & 0 \end{bmatrix}\begin{Bmatrix} \theta_1^{(n)} \\ \theta_2^{(n)} \\ \theta_3^{(n)} \end{Bmatrix} \quad (14)$$

Strain Calculation 3

- If each component is calculated, the following should be obtained.

$$\frac{\partial u_1}{\partial r_1} = \tag{15}$$

$$\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left[u_1^{(n)} + \left(\frac{a}{2}r_2V_{23}^{(n)} + \frac{b}{2}r_3V_{33}^{(n)}\right)\theta_2^{(n)} + \left(-\frac{a}{2}r_2V_{22}^{(n)} - \frac{b}{2}r_3V_{32}^{(n)}\right)\theta_3^{(n)}\right] \tag{16}$$

$$\frac{\partial u_2}{\partial r_1} = \tag{17}$$

$$\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left[u_2^{(n)} + \left(-\frac{a}{2}r_2V_{23}^{(n)} - \frac{b}{2}r_3V_{33}^{(n)}\right)\theta_1^{(n)} + \left(\frac{a}{2}r_2V_{21}^{(n)} - \frac{b}{2}r_3V_{31}^{(n)}\right)\theta_3^{(n)}\right] \tag{18}$$

$$\frac{\partial u_3}{\partial r_1} = \tag{19}$$

$$\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left[u_3^{(n)} + \left(\frac{a}{2}r_2V_{22}^{(n)} + \frac{b}{2}r_3V_{32}^{(n)}\right)\theta_1^{(n)} + \left(-\frac{a}{2}r_2V_{21}^{(n)} - \frac{b}{2}r_3V_{31}^{(n)}\right)\theta_2^{(n)}\right] \tag{20}$$

$$\frac{\partial u_1}{\partial r_2} = \frac{a}{2}N^{(n)}(r_1)V_{23}^{(n)}\theta_2^{(n)} - \frac{a}{2}N^{(n)}(r_1)V_{22}^{(n)}\theta_3^{(n)} \tag{21}$$

$$\frac{\partial u_2}{\partial r_2} = -\frac{a}{2}N^{(n)}(r_1)V_{23}^{(n)}\theta_1^{(n)} + \frac{a}{2}N^{(n)}(r_1)V_{21}^{(n)}\theta_3^{(n)} \tag{22}$$

$$\frac{\partial u_3}{\partial r_2} = \frac{a}{2}N^{(n)}(r_1)V_{22}^{(n)}\theta_1^{(n)} - \frac{a}{2}N^{(n)}(r_1)V_{21}^{(n)}\theta_2^{(n)} \tag{23}$$

$$\frac{\partial u_1}{\partial r_3} = \frac{b}{2}N^{(n)}(r_1)V_{33}^{(n)}\theta_2^{(n)} - \frac{b}{2}N^{(n)}(r_1)V_{32}^{(n)}\theta_3^{(n)} \tag{24}$$

$$\frac{\partial u_2}{\partial r_3} = -\frac{b}{2}N^{(n)}(r_1)V_{33}^{(n)}\theta_1^{(n)} + \frac{b}{2}N^{(n)}(r_1)V_{31}^{(n)}\theta_3^{(n)} \tag{25}$$

$$\frac{\partial u_3}{\partial r_3} = \frac{b}{2}N^{(n)}(r_1)V_{32}^{(n)}\theta_1^{(n)} - \frac{b}{2}N^{(n)}(r_1)V_{31}^{(n)}\theta_2^{(n)} \tag{26}$$

Strain Calculation 4

- To organize the expression, modify the index .

$$\frac{\partial u_i}{\partial x_j} \Longrightarrow \frac{\partial u_j}{\partial x_i} = J_{ik}^{-1} \frac{\partial u_j}{\partial r_k} \tag{27}$$

- Calculate each component,

$$\frac{\partial u_1}{\partial x_i} = J_{i1}^{-1} \frac{\partial u_1}{\partial r_1} + J_{i2}^{-1} \frac{\partial u_1}{\partial r_2} + J_{i3}^{-1} \frac{\partial u_1}{\partial r_3} \tag{28}$$

$$\begin{aligned} &= J_{i1}^{-1} \frac{\partial N^{(n)}(r_1)}{\partial r_1} \left\{ u_1^{(n)} + \left(\frac{a}{2} r_2 V_{23}^{(n)} + \frac{b}{2} r_3 V_{23}^{(n)} \right) \theta_2^{(n)} + \left(-\frac{a}{2} r_2 V_{22}^{(n)} - \frac{b}{2} r_3 V_{32}^{(n)} \right) \theta_3^{(n)} \right\} \\ &+ J_{i2}^{-1} \left\{ \frac{a}{2} N^{(n)}(r_1) V_{22}^{(n)} \theta_2^{(n)} - \frac{a}{2} N^{(n)}(r_1) V_{22}^{(n)} \theta_3^{(n)} \right\} \\ &+ J_{i3}^{-1} \left\{ \frac{b}{2} N^{(n)}(r_1) V_{33}^{(n)} \theta_2^{(n)} - \frac{b}{2} N^{(n)}(r_1) V_{32}^{(n)} \theta_3^{(n)} \right\} \end{aligned} \tag{29}$$

$$\begin{aligned} &= J_{i1}^{-1} \frac{\partial N^{(n)}(r_1)}{\partial r_1} u_1^{(n)} \\ &+ \left\{ J_{i1}^{-1} \frac{\partial N^{(n)}(r_1)}{\partial r_1} \left(\frac{a}{2} r_2 V_{23}^{(n)} + \frac{b}{2} r_3 V_{33}^{(n)} \right) + J_{i2}^{-1} \frac{a}{2} N^{(n)}(r_1) V_{23}^{(n)} + J_{i3}^{-1} \frac{b}{2} N^{(n)}(r_1) V_{33}^{(n)} \right\} \theta_2^{(n)} \\ &+ \left\{ J_{i1}^{-1} \frac{\partial N^{(n)}(r_1)}{\partial r_1} \left(-\frac{a}{2} r_2 V_{22}^{(n)} - \frac{b}{2} r_3 V_{32}^{(n)} \right) - J_{i2}^{-1} \frac{a}{2} N^{(n)}(r_1) V_{22}^{(n)} - J_{i3}^{-1} \frac{b}{2} N^{(n)}(r_1) V_{32}^{(n)} \right\} \theta_3^{(n)} \end{aligned} \tag{30}$$

Here , the coefficient for rotational components $\theta_m^{(n)}$ in $\partial u_1 / \partial x_i$ is expressed by $\left(G_m^{(n)}\right)_{1i}$.
Thus,

$$\left(G_1^{(n)}\right)_{1i} = 0 \tag{31}$$

$$\left(G_2^{(n)}\right)_{1i} = J_{i1}^{-1} \frac{\partial N^{(n)}(r_1)}{\partial r_1} \left(\frac{a}{2} r_2 V_{23}^{(n)} + \frac{b}{2} r_3 V_{33}^{(n)} \right) + J_{i2}^{-1} \frac{a}{2} N^{(n)}(r_1) V_{23}^{(n)} + J_{i3}^{-1} \frac{b}{2} N^{(n)}(r_1) V_{33}^{(n)} \tag{32}$$

$$\left(G_3^{(n)}\right)_{1i} = J_{i1}^{-1} \frac{\partial N^{(n)}(r_1)}{\partial r_1} \left(-\frac{a}{2} r_2 V_{22}^{(n)} - \frac{b}{2} r_3 V_{32}^{(n)} \right) - J_{i2}^{-1} \frac{a}{2} N^{(n)}(r_1) V_{22}^{(n)} - J_{i3}^{-1} \frac{b}{2} N^{(n)}(r_1) V_{32}^{(n)} \tag{33}$$

$$
\frac{\partial u_2}{\partial x_i} = J_{i1}^{-1}\frac{\partial u_2}{\partial r_1} + J_{i2}^{-1}\frac{\partial u_2}{\partial r_2} + J_{i3}^{-1}\frac{\partial u_2}{\partial r_3} \tag{34}
$$

$$
\begin{aligned}
&= J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left\{u_2^{(n)} + \left(-\frac{a}{2}r_2V_{23}^{(n)} - \frac{b}{2}r_3V_{33}^{(n)}\right)\theta_1^{(n)} + \left(\frac{a}{2}r_2V_{21}^{(n)} + \frac{b}{2}r_3V_{31}^{(n)}\right)\theta_3^{(n)}\right\} \\
&+ J_{i2}^{-1}\left\{-\frac{a}{2}N^{(n)}(r_1)V_{23}^{(n)}\theta_1^{(n)} + \frac{a}{2}N^{(n)}(r_1)V_{21}^{(n)}\theta_3^{(n)}\right\} \\
&+ J_{i3}^{-1}\left\{-\frac{b}{2}N^{(n)}(r_1)V_{33}^{(n)}\theta_1^{(n)} + \frac{b}{2}N^{(n)}(r_1)V_{31}^{(n)}\theta_3^{(n)}\right\}
\end{aligned} \tag{35}
$$

$$
\begin{aligned}
&= J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}u_2^{(n)} \\
&+ \left\{J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left(-\frac{a}{2}r_2V_{23}^{(n)} - \frac{b}{2}r_3V_{32}^{(n)}\right) - J_{i2}^{-1}\frac{a}{2}N^{(n)}(r_1)V_{23}^{(n)} - J_{i3}^{-1}\frac{b}{2}N^{(n)}(r_1)V_{33}^{(n)}\right\}\theta_1^{(n)} \\
&+ \left\{J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left(\frac{a}{2}r_2V_{21}^{(n)} + \frac{b}{2}r_3V_{31}^{(n)}\right) + J_{i2}^{-1}\frac{a}{2}N^{(n)}(r_1)V_{21}^{(n)} + J_{i3}^{-1}\frac{b}{2}N^{(n)}(r_1)V_{31}^{(n)}\right\}\theta_3^{(n)}
\end{aligned} \tag{36}
$$

$$
\left(G_1^{(n)}\right)_{2i} = J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left(-\frac{a}{2}r_2V_{23}^{(n)} - \frac{b}{2}r_3V_{32}^{(n)}\right) - J_{i2}^{-1}\frac{a}{2}N^{(n)}(r_1)V_{23}^{(n)} - J_{i3}^{-1}\frac{b}{2}N^{(n)}(r_1)V_{33}^{(n)} \tag{37}
$$

$$
\left(G_2^{(n)}\right)_{2i} = 0 \tag{38}
$$

$$
\left(G_3^{(n)}\right)_{2i} = J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left(\frac{a}{2}r_2V_{21}^{(n)} + \frac{b}{2}r_3V_{31}^{(n)}\right) + J_{i2}^{-1}\frac{a}{2}N^{(n)}(r_1)V_{21}^{(n)} + J_{i3}^{-1}\frac{b}{2}N^{(n)}(r_1)V_{31}^{(n)} \tag{39}
$$

$$
\begin{aligned}
\frac{\partial u_3}{\partial x_i} &= J_{i1}^{-1}\frac{\partial u_3}{\partial r_1} + J_{i2}^{-1}\frac{\partial u_3}{\partial r_2} + J_{i3}^{-1}\frac{\partial u_3}{\partial r_3} &(40)\\
&= J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left\{u_3^{(n)} + \left(\frac{a}{2}r_2V_{22}^{(n)} + \frac{b}{2}r_3V_{32}^{(n)}\right)\theta_1^{(n)} + \left(-\frac{a}{2}r_2V_{21}^{(n)} - \frac{b}{2}r_3V_{31}^{(n)}\right)\theta_2^{(n)}\right\}\\
&+ J_{i2}^{-1}\left\{\frac{a}{2}N^{(n)}(r_1)V_{22}^{(n)}\theta_1^{(n)} - \frac{a}{2}N^{(n)}(r_1)V_{21}^{(n)}\theta_2^{(n)}\right\}\\
&+ J_{i3}^{-1}\left\{\frac{b}{2}N^{(n)}(r_1)V_{32}^{(n)}\theta_1^{(n)} - \frac{b}{2}N^{(n)}(r_1)V_{31}^{(n)}\theta_2^{(n)}\right\} &(41)\\
&= J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}u_3^{(n)}\\
&+ \left\{J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left(\frac{a}{2}r_2V_{22}^{(n)} + \frac{b}{2}r_3V_{32}^{(n)}\right) + J_{i2}^{-1}\frac{a}{2}N^{(n)}(r_1)V_{22}^{(n)} + J_{i3}^{-1}\frac{b}{2}N^{(n)}(r_1)V_{32}^{(n)}\right\}\theta_1^{(n)} &(42)\\
&+ \left\{J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left(-\frac{a}{2}r_2V_{21}^{(n)} - \frac{b}{2}r_3V_{31}^{(n)}\right) - J_{i2}^{-1}\frac{a}{2}N^{(n)}(r_1)V_{21}^{(n)} - J_{i3}^{-1}\frac{b}{2}N^{(n)}(r_1)V_{31}^{(n)}\right\}\theta_2^{(n)}
\end{aligned}
$$

Here in the same way,express the coefficient for the rotational component $\theta_m^{(n)}$ in $\partial u_3/\partial x_i$ as $\left(G_m^{(n)}\right)_{3i}$.

$$
\begin{aligned}
\left(G_1^{(n)}\right)_{3i} &= J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left(\frac{a}{2}r_2V_{22}^{(n)} + \frac{b}{2}r_3V_{32}^{(n)}\right) + J_{i2}^{-1}\frac{a}{2}N^{(n)}(r_1)V_{22}^{(n)} + J_{i3}^{-1}\frac{b}{2}N^{(n)}(r_1)V_{32}^{(n)} &(43)\\
\left(G_2^{(n)}\right)_{3i} &= J_{i1}^{-1}\frac{\partial N^{(n)}(r_1)}{\partial r_1}\left(-\frac{a}{2}r_2V_{21}^{(n)} - \frac{b}{2}r_3V_{31}^{(n)}\right) - J_{i2}^{-1}\frac{a}{2}N^{(n)}(r_1)V_{21}^{(n)} - J_{i3}^{-1}\frac{b}{2}N^{(n)}(r_1)V_{31}^{(n)} &(44)\\
\left(G_3^{(n)}\right)_{3i} &= 0 &(45)
\end{aligned}
$$

Strain Calculation 5

- In summary, the infinitesimal strain can be expressed in the following matrix.

$$
\begin{bmatrix} E_{11} \\ E_{22} \\ E_{33} \\ 2E_{12} \\ 2E_{23} \\ 2E_{31} \end{bmatrix} = \begin{bmatrix} \frac{\partial u_1}{\partial x_1} \\ \frac{\partial u_2}{\partial x_2} \\ \frac{\partial u_3}{\partial x_3} \\ \frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1} \\ \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} \\ \frac{\partial u_3}{\partial x_1} + \frac{\partial u_1}{\partial x_3} \end{bmatrix} = \left[\bar{B}\right] \left\{u^{(n)}\right\} \tag{46}
$$

Provided that,

$$
\left\{u^{(n)}\right\} = \left\{u_1^{(1)}\ u_2^{(1)}\ u_3^{(1)}\ \theta_1^{(1)}\ \theta_2^{(1)}\ \theta_3^{(1)}\ \cdots\ u_1^{(n)}\ u_2^{(n)}\ u_3^{(n)}\ \theta_1^{(n)}\ \theta_2^{(n)}\ \theta_3^{(n)}\right\} \tag{47}
$$

$$
\left[\bar{B}\right] = \left[\left[\bar{B}_1\right]\ \left[\bar{B}_2\right]\ \cdots\ \left[\bar{B}_n\right]\right] \tag{48}
$$

$$
\left[\bar{B}_n\right] = \begin{bmatrix}
J_{11}^{-1}\frac{\partial N^{(n)}}{\partial r_1} & 0 & 0 & 0 & (G2)_{11} & (G3)_{11} \\
0 & J_{21}^{-1}\frac{\partial N^{(n)}}{\partial r_1} & 0 & (G1)_{22} & 0 & (G3)_{22} \\
0 & 0 & J_{31}^{-1}\frac{\partial N^{(n)}}{\partial r_1} & (G1)_{33} & (G2)_{33} & 0 \\
0 & 0 & J_{31}^{-1}\frac{\partial N^{(n)}}{\partial r_1} & (G1)_{33} & (G2)_{33} & 0 \\
J_{21}^{-1}\frac{\partial N^{(n)}}{\partial r_1} & J_{11}^{-1}\frac{\partial N^{(n)}}{\partial r_1} & 0 & (G1)_{21} & (G2)_{12} & (G3)_{12} + (G3)_{21} \\
0 & J_{31}^{-1}\frac{\partial N^{(n)}}{\partial r_1} & J_{21}^{-1}\frac{\partial N^{(n)}}{\partial r_1} & (G1)_{23} + (G1)_{32} & (G2)_{32} & (G3)_{23} \\
J_{31}^{-1}\frac{\partial N^{(n)}}{\partial r_1} & 0 & J_{11}^{-1}\frac{\partial N^{(n)}}{\partial r_1} & (G1)_{31} & (G2)_{31} + (G2)_{13} & (G3)_{13}
\end{bmatrix} \tag{49}
$$

Strain Calculation 6

- In formulation by Total-Lagrane method, term $\frac{\partial {}^t u_i}{\partial X_j}$ may appear in Green-Lagrenge strain.

$$
\begin{aligned}
\frac{\partial {}^t u_i}{\partial X_j} &= \frac{\partial N^{(n)}}{\partial X_j} {}^t u_i^{(n)} \\
&\quad + \frac{a}{2} \frac{\partial r_2 N^{(n)}}{\partial X_j} \left({}^t \boldsymbol{V}_{2i}^{(n)} - {}^0 \boldsymbol{V}_{2i}^{(n)} \right) \\
&\quad + \frac{b}{2} \frac{\partial r_3 N^{(n)}}{\partial X_j} \left({}^t \boldsymbol{V}_{3i}^{(n)} - {}^0 \boldsymbol{V}_{3i}^{(n)} \right)
\end{aligned} \tag{50}
$$

Solid Element 1

- δE_{ij} can be expressed by components,

$$\delta E_{ij} = \frac{1}{2}\left(\frac{\partial \delta u_i}{\partial X_j} + \frac{\partial \delta u_j}{\partial X_i} + \frac{\partial \delta u_k}{\partial X_i}\frac{\partial u_k}{\partial X_j} + \frac{\partial u_k}{\partial X_i}\frac{\partial \delta u_k}{\partial X_j}\right)$$

$$\begin{aligned}
\delta E_{11} &= \frac{1}{2}\left(\frac{\partial \delta u_1}{\partial X_1} + \frac{\partial \delta u_1}{\partial X_1} + \frac{\partial \delta u_1}{\partial X_1}\frac{\partial u_1}{\partial X_1} + \frac{\partial u_1}{\partial X_1}\frac{\partial \delta u_1}{\partial X_1} + \frac{\partial \delta u_2}{\partial X_1}\frac{\partial u_2}{\partial X_1} + \frac{\partial u_2}{\partial X_1}\frac{\partial \delta u_2}{\partial X_1} + \frac{\partial \delta u_3}{\partial X_1}\frac{\partial u_3}{\partial X_1} + \frac{\partial u_3}{\partial X_1}\frac{\partial \delta u_3}{\partial X_1}\right) \\
\delta E_{22} &= \frac{1}{2}\left(\frac{\partial \delta u_2}{\partial X_2} + \frac{\partial \delta u_2}{\partial X_2} + \frac{\partial \delta u_1}{\partial X_2}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_1}{\partial X_2}\frac{\partial \delta u_1}{\partial X_2} + \frac{\partial \delta u_2}{\partial X_2}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_2}{\partial X_2}\frac{\partial \delta u_2}{\partial X_2} + \frac{\partial \delta u_3}{\partial X_2}\frac{\partial u_3}{\partial X_2} + \frac{\partial u_3}{\partial X_2}\frac{\partial \delta u_3}{\partial X_2}\right) \\
\delta E_{33} &= \frac{1}{2}\left(\frac{\partial \delta u_3}{\partial X_3} + \frac{\partial \delta u_3}{\partial X_3} + \frac{\partial \delta u_1}{\partial X_3}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_1}{\partial X_3}\frac{\partial \delta u_1}{\partial X_3} + \frac{\partial \delta u_2}{\partial X_3}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_2}{\partial X_3}\frac{\partial \delta u_2}{\partial X_3} + \frac{\partial \delta u_3}{\partial X_3}\frac{\partial u_3}{\partial X_3} + \frac{\partial u_3}{\partial X_3}\frac{\partial \delta u_3}{\partial X_3}\right) \\
\delta E_{12} &= \frac{1}{2}\left(\frac{\partial \delta u_1}{\partial X_2} + \frac{\partial \delta u_2}{\partial X_1} + \frac{\partial \delta u_1}{\partial X_1}\frac{\partial u_1}{\partial X_2} + \frac{\partial u_1}{\partial X_1}\frac{\partial \delta u_1}{\partial X_2} + \frac{\partial \delta u_2}{\partial X_1}\frac{\partial u_2}{\partial X_2} + \frac{\partial u_2}{\partial X_1}\frac{\partial \delta u_2}{\partial X_2} + \frac{\partial \delta u_3}{\partial X_1}\frac{\partial u_3}{\partial X_2} + \frac{\partial u_3}{\partial X_1}\frac{\partial \delta u_3}{\partial X_2}\right) \\
\delta E_{23} &= \frac{1}{2}\left(\frac{\partial \delta u_2}{\partial X_3} + \frac{\partial \delta u_3}{\partial X_2} + \frac{\partial \delta u_1}{\partial X_2}\frac{\partial u_1}{\partial X_3} + \frac{\partial u_1}{\partial X_2}\frac{\partial \delta u_1}{\partial X_3} + \frac{\partial \delta u_2}{\partial X_2}\frac{\partial u_2}{\partial X_3} + \frac{\partial u_2}{\partial X_2}\frac{\partial \delta u_2}{\partial X_3} + \frac{\partial \delta u_3}{\partial X_2}\frac{\partial u_3}{\partial X_3} + \frac{\partial u_3}{\partial X_2}\frac{\partial \delta u_3}{\partial X_3}\right) \\
\delta E_{31} &= \frac{1}{2}\left(\frac{\partial \delta u_3}{\partial X_1} + \frac{\partial \delta u_1}{\partial X_3} + \frac{\partial \delta u_1}{\partial X_3}\frac{\partial u_1}{\partial X_1} + \frac{\partial u_1}{\partial X_3}\frac{\partial \delta u_1}{\partial X_1} + \frac{\partial \delta u_2}{\partial X_3}\frac{\partial u_2}{\partial X_1} + \frac{\partial u_2}{\partial X_3}\frac{\partial \delta u_2}{\partial X_1} + \frac{\partial \delta u_3}{\partial X_3}\frac{\partial u_3}{\partial X_1} + \frac{\partial u_3}{\partial X_3}\frac{\partial \delta u_3}{\partial X_1}\right)
\end{aligned}$$

- Thus,

$$[Z_1] \equiv$$

$$\begin{bmatrix}
1+\frac{\partial u_1}{\partial X_1} & 0 & 0 & \frac{\partial u_2}{\partial X_1} & 0 & 0 & \frac{\partial u_3}{\partial X_1} & 0 & 0 \\
0 & \frac{\partial u_1}{\partial X_2} & 0 & 0 & 1+\frac{\partial u_2}{\partial X_2} & 0 & 0 & \frac{\partial u_3}{\partial X_2} & 0 \\
0 & 0 & \frac{\partial u_1}{\partial X_3} & 0 & 0 & \frac{\partial u_2}{\partial X_3} & 0 & 0 & 1+\frac{\partial u_3}{\partial X_3} \\
\frac{\partial u_1}{\partial X_2} & 1+\frac{\partial u_1}{\partial X_1} & 0 & 1+\frac{\partial u_2}{\partial X_2} & \frac{\partial u_2}{\partial X_1} & 0 & \frac{\partial u_3}{\partial X_2} & \frac{\partial u_3}{\partial X_1} & 0 \\
0 & \frac{\partial u_1}{\partial X_3} & \frac{\partial u_1}{\partial X_2} & 0 & \frac{\partial u_2}{\partial X_3} & 1+\frac{\partial u_2}{\partial X_2} & 0 & 1+\frac{\partial u_3}{\partial X_3} & \frac{\partial u_3}{\partial X_2} \\
\frac{\partial u_1}{\partial X_3} & 0 & 1+\frac{\partial u_1}{\partial X_1} & \frac{\partial u_2}{\partial X_3} & 0 & \frac{\partial u_2}{\partial X_1} & 1+\frac{\partial u_3}{\partial X_3} & 0 & \frac{\partial u_3}{\partial X_1}
\end{bmatrix}$$

$$\left\{\frac{\partial \delta u}{\partial X}\right\} \equiv \left\{\frac{\partial \delta u_1}{\partial X_1}\ \frac{\partial \delta u_1}{\partial X_2}\ \frac{\partial \delta u_1}{\partial X_3}\ \frac{\partial \delta u_2}{\partial X_1}\ \frac{\partial \delta u_2}{\partial X_2}\ \frac{\partial \delta u_2}{\partial X_3}\ \frac{\partial \delta u_3}{\partial X_1}\ \frac{\partial \delta u_3}{\partial X_2}\ \frac{\partial \delta u_3}{\partial X_3}\right\}^T$$

$$\{\delta E\} = [Z_1] \left\{ \frac{\partial \delta u}{\partial X} \right\}$$

- For $\frac{\partial u_i}{\partial X_j}$ appears in this matrix, obtained by the following equation.

$$\frac{\partial u_i}{\partial X_j} = \frac{\partial N^{(n)}}{\partial X_j} u_i^{(n)}$$

Solid Element 2

- In the same way, $\frac{\partial \delta u_i}{\partial X_j}$ can be written as,

$$\frac{\partial \delta u_i}{\partial X_j} = \frac{\partial N^{(n)}}{\partial X_j}\delta u_i^{(n)}$$

- From the equation above, $\left\{\frac{\partial \delta u}{\partial X}\right\}$ can be expressed in node deformation.

$$\left\{\begin{array}{c} \frac{\partial \delta u_1}{\partial X_1} \\ \frac{\partial \delta u_1}{\partial X_2} \\ \frac{\partial \delta u_1}{\partial X_3} \\ \frac{\partial \delta u_2}{\partial X_1} \\ \frac{\partial \delta u_2}{\partial X_2} \\ \frac{\partial \delta u_2}{\partial X_3} \\ \frac{\partial \delta u_3}{\partial X_1} \\ \frac{\partial \delta u_3}{\partial X_2} \\ \frac{\partial \delta u_3}{\partial X_3} \end{array}\right\} = \left[\begin{array}{ccccccc} \frac{\partial N^{(1)}}{\partial X_1} & & & & \frac{\partial N^{(n)}}{\partial X_1} & & \\ \frac{\partial N^{(1)}}{\partial X_2} & & & & \frac{\partial N^{(n)}}{\partial X_2} & & \\ \frac{\partial N^{(1)}}{\partial X_3} & & & & \frac{\partial N^{(n)}}{\partial X_3} & & \\ & \frac{\partial N^{(1)}}{\partial X_1} & & & & \frac{\partial N^{(n)}}{\partial X_1} & \\ & \frac{\partial N^{(1)}}{\partial X_2} & & \cdots & & \frac{\partial N^{(n)}}{\partial X_2} & \\ & \frac{\partial N^{(1)}}{\partial X_3} & & & & \frac{\partial N^{(n)}}{\partial X_3} & \\ & & \frac{\partial N^{(1)}}{\partial X_1} & & & & \frac{\partial N^{(n)}}{\partial X_1} \\ & & \frac{\partial N^{(1)}}{\partial X_2} & & & & \frac{\partial N^{(n)}}{\partial X_2} \\ & & \frac{\partial N^{(1)}}{\partial X_3} & & & & \frac{\partial N^{(n)}}{\partial X_3} \end{array}\right] \left\{\begin{array}{c} \delta u_1^{(1)} \\ \delta u_2^{(1)} \\ \delta u_3^{(1)} \\ \vdots \\ \delta u_1^{(n)} \\ \delta u_2^{(n)} \\ \delta u_3^{(n)} \end{array}\right\}$$

- Define the above $9 \text{ x} 3n$ matrix $[Z_2]$ to be the following.

$$\left\{ \frac{\partial \delta u}{\partial X} \right\} = [Z_2]\{\delta u^{(n)}\}$$

Comparison With Solid Element

- Solid element of $[Z_2]$

$$[Z_2] = \begin{bmatrix} \frac{\partial N}{\partial X_1} & 0 & 0 \\ \frac{\partial N}{\partial X_2} & 0 & 0 \\ \frac{\partial N}{\partial X_3} & 0 & 0 \\ 0 & \frac{\partial N}{\partial X_1} & 0 \\ 0 & \frac{\partial N}{\partial X_2} & 0 \\ 0 & \frac{\partial N}{\partial X_3} & 0 \\ 0 & 0 & \frac{\partial N}{\partial X_1} \\ 0 & 0 & \frac{\partial N}{\partial X_2} \\ 0 & 0 & \frac{\partial N}{\partial X_3} \end{bmatrix}$$

- Beam element of $[Z_2]$

$$
[Z_2] = \begin{bmatrix}
\frac{\partial N}{\partial X_1} & 0 & 0 & 0 & (G_2)_{11} & (G_3)_{11} \\
\frac{\partial N}{\partial X_2} & 0 & 0 & 0 & (G_2)_{12} & (G_3)_{12} \\
\frac{\partial N}{\partial X_3} & 0 & 0 & 0 & (G_2)_{13} & (G_3)_{13} \\
0 & \frac{\partial N}{\partial X_1} & 0 & (G_1)_{21} & 0 & (G_3)_{21} \\
0 & \frac{\partial N}{\partial X_2} & 0 & (G_1)_{22} & 0 & (G_3)_{22} \\
0 & \frac{\partial N}{\partial X_3} & 0 & (G_1)_{23} & 0 & (G_3)_{23} \\
0 & 0 & \frac{\partial N}{\partial X_1} & (G_1)_{31} & (G_2)_{31} & 0 \\
0 & 0 & \frac{\partial N}{\partial X_2} & (G_1)_{32} & (G_2)_{32} & 0 \\
0 & 0 & \frac{\partial N}{\partial X_3} & (G_1)_{33} & (G_2)_{33} & 0
\end{bmatrix}
$$