GLOBAL FOCUS ON KNOWLEDGE
"CREATING MATHEMATICS" －THE CHALLENGE OF MATHEMATICIANS－

2009 winter semester

Lecture 1 "How Mathematics Was Created"

2009/10/08 Kazuo Murota⊕Kazuo Okamoto

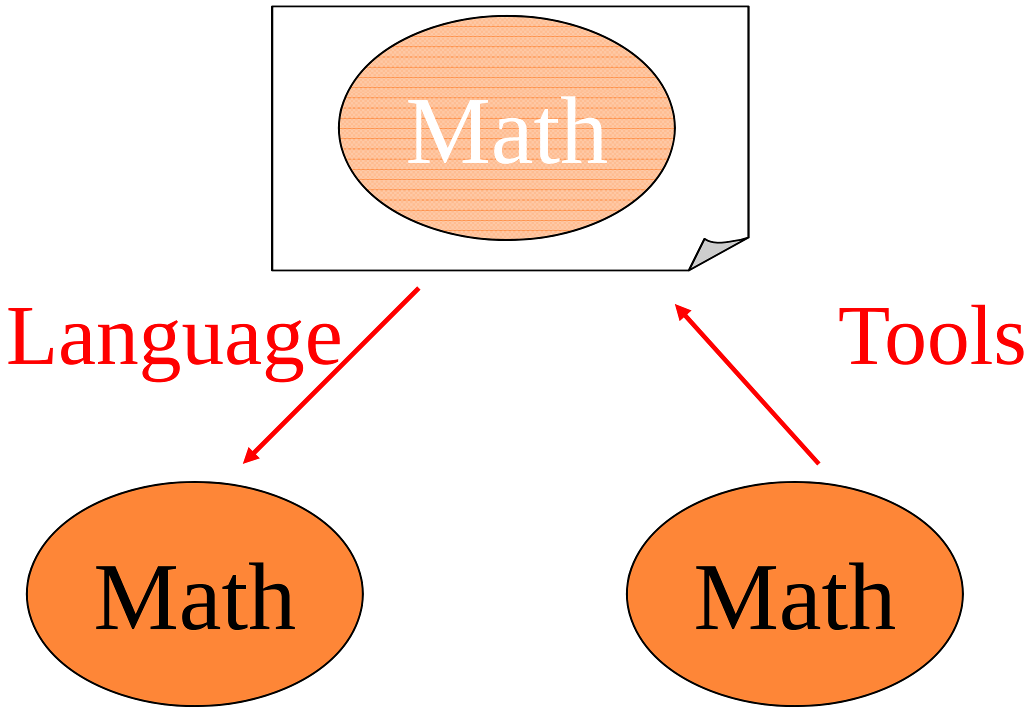
WHAT IS MATHEMATICS?

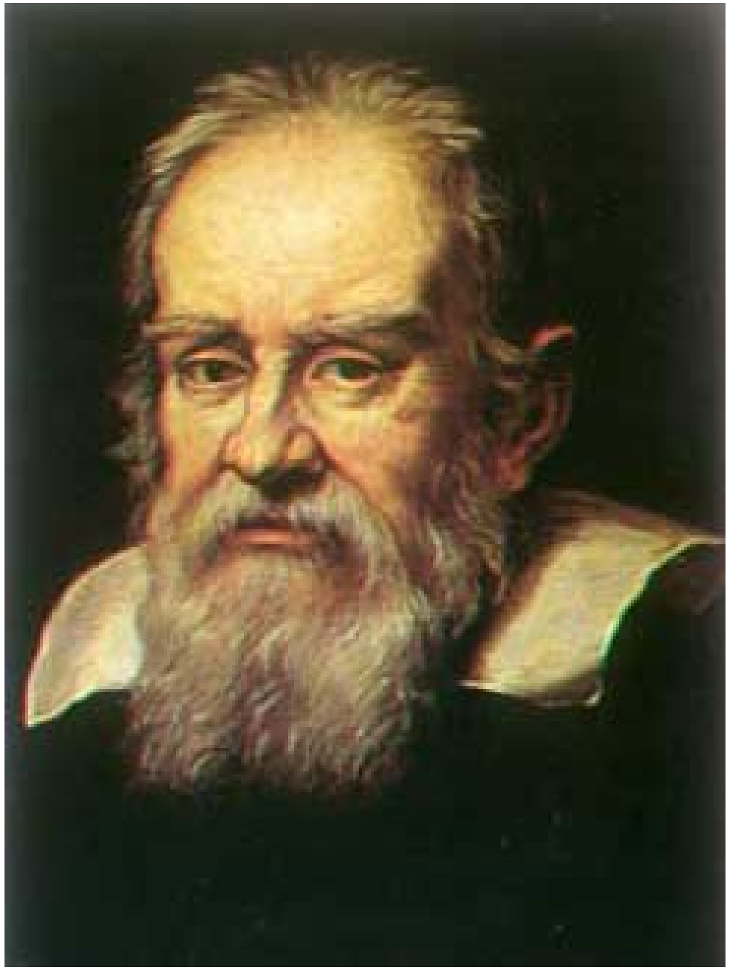
- What does it deal with?
- Where has it been brought up?
- What does it originate from?

THREE FORMS OF MATHEMATICS

- Sugaku (math), Sugaku (enjoy math), Sugaku (suffer from math)
- mathematics as a language
- mathematics as a tool
- mathematics as an object

“Create Mathematics”?





“Mathematics is the language with which God has written the universe“

Galileo Galilei
1564-1642

INCENTIVES TO CREATE MATHEMATICS

When we want to

- discover something,
- solve some problem,
- or simply use math

- Incentives to create new mathematics lie in nature, society, or math itself
- “Necessity is the mother of invention”, but playful mind is also important
- Change the point of view

$$0+4=$$

$$3+4=$$

$$3\times 2+4=$$

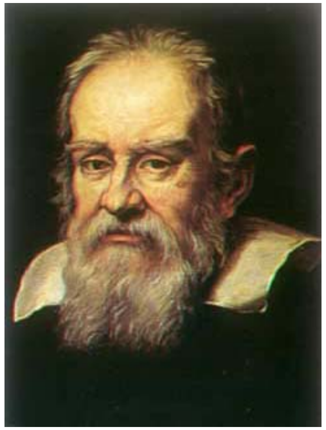
$$3\times 2^2+4=$$

$$3\times 2^3+4=$$

$$3\times 2^4+4=$$

$$3\times 2^5+4=$$

Sequence calculation



$$0+4=4,$$

$$3+4=7,$$

$$3\times 2+4=10$$

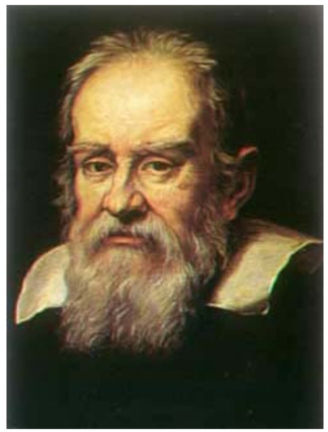
$$3\times 2^{2}+4=16,$$

$$3\times 2^{3}+4=28,$$

$$3\times 2^{4}+4=52,$$

$$3\times 2^{5}+4=100$$

Good work

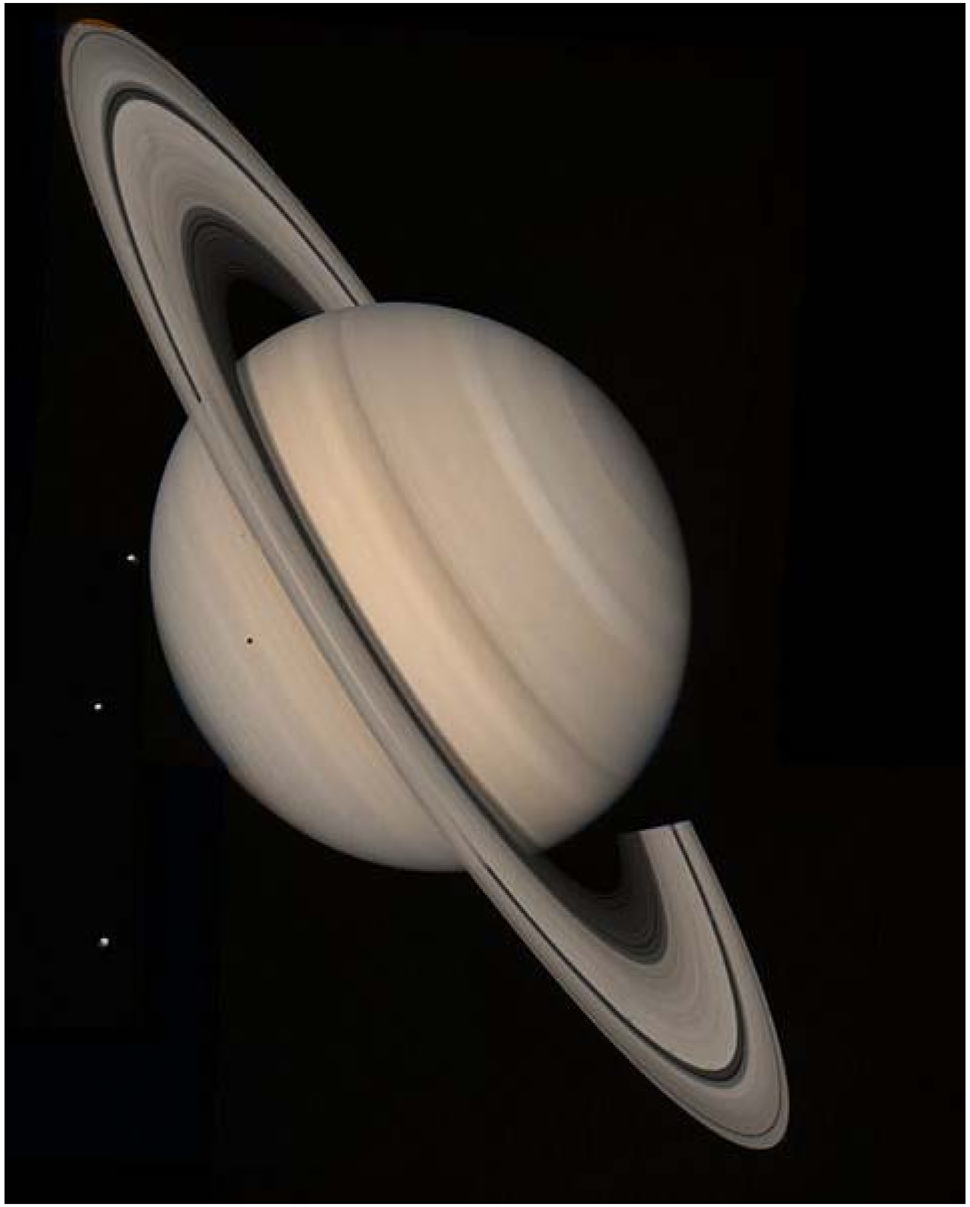


Bode's law

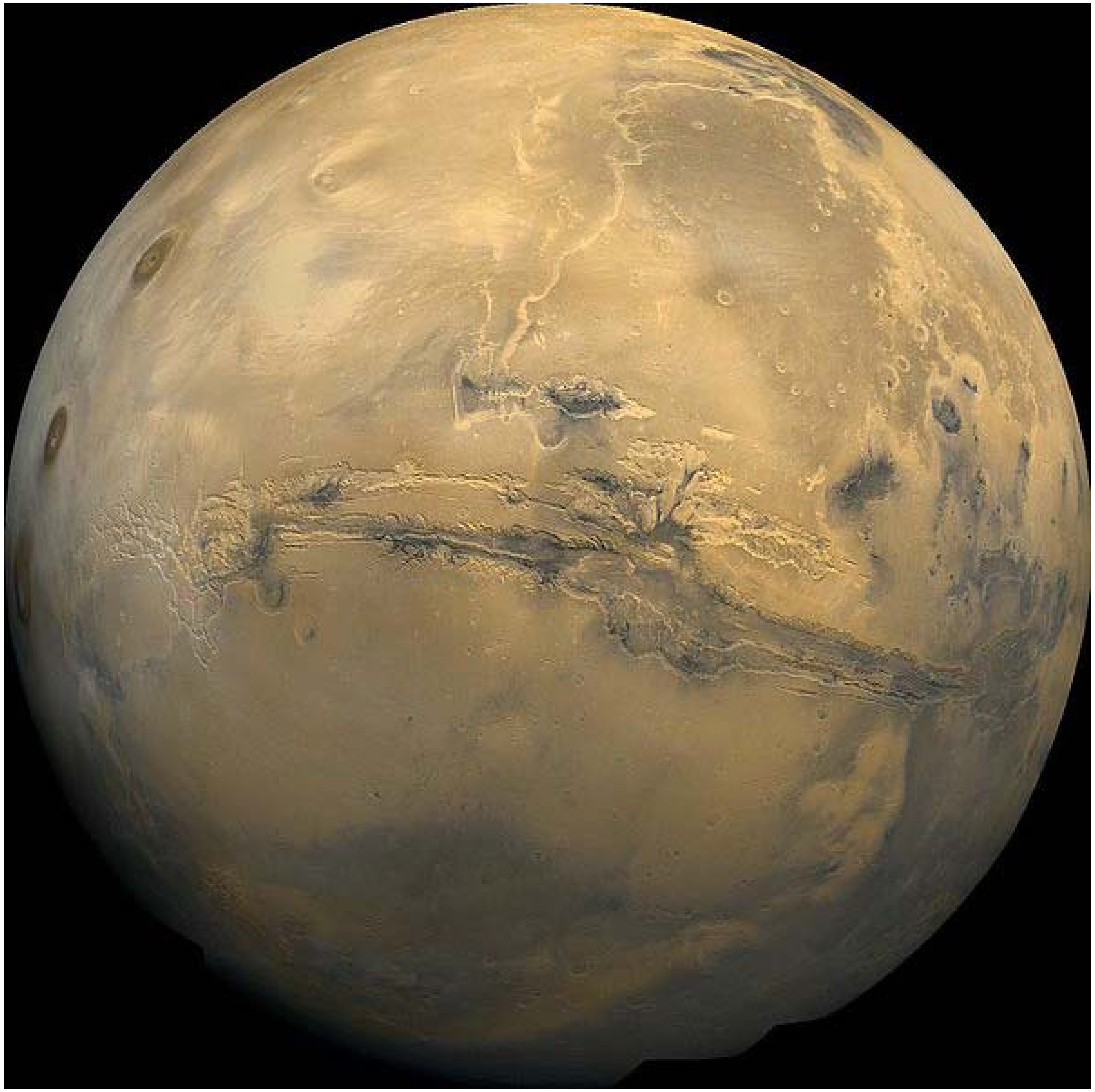
Average distance between planets and the sun

sun-earth distance: 10

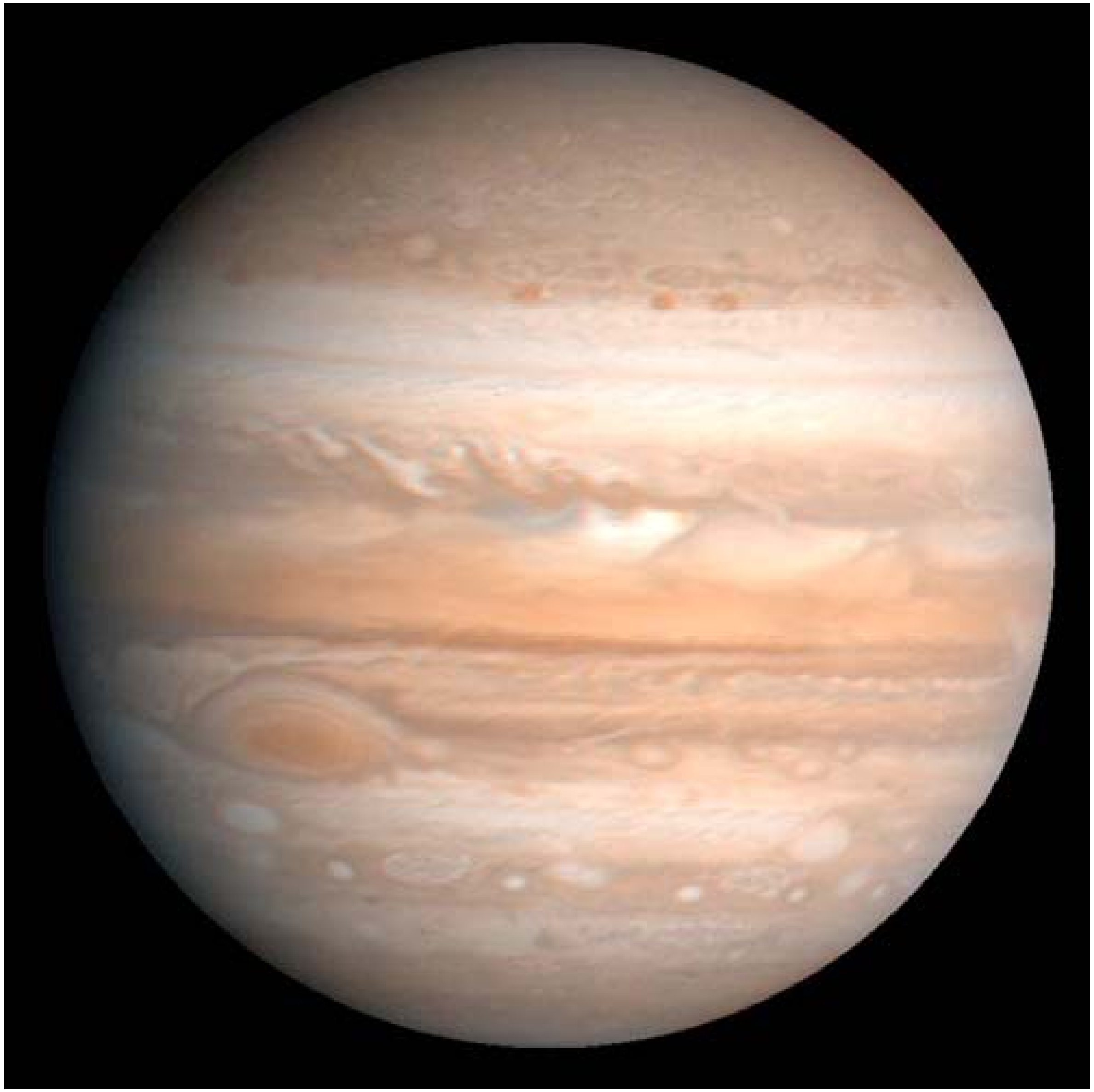
Mercury : 3.9   Venus : 7.2   Earth : 10
Mars : 15.2   Jupiter : 52   Saturn : 95.4



95.4



15.2



52

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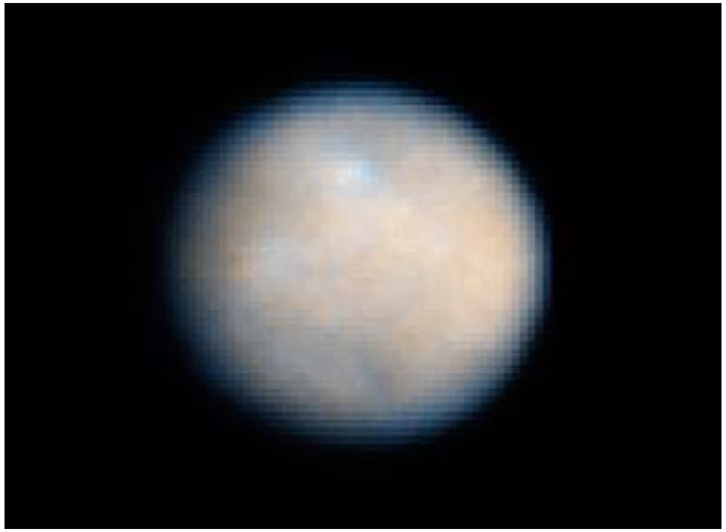
Mercury : 3.9

Venus : 7.2 Earth : 10

Mars : 15.2 Jupiter : 52 Saturn : 95.4

DISCOVERIES OF ASTEROIDS

- **28** is lacking!
- In **1801**, Asteroid Ceres(**27.7**) discovered
- And lost



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Reprinted from
http://en.wikipedia.org/wiki/File:Bendixen_-_Carl_Friedrich_Gauß,_1828.jpg (2010/03/02)

Carl Friedrich Gauss
1777-1855

CARL FRIEDRICH GAUSS, DIRECTOR OF GETTINGEN ASTRONOMICAL OBSERVATORY

- Orbital calculation and the rediscovery of Ceres
- Least-square method adopted
- A new theory born

Global Focus on Knowledge

Creating Mathematics

-the Challenge of Mathematics-

(1st lecture) 2009-10-08

How mathematics was created

The mathematics for engineering and optimization

Kazuo Murota

Dept. of Mathematical Engineering and Information Physics, Faculty of Engineering

Dept. of Mathematical Informatics, Graduate School of Information Science and Technology

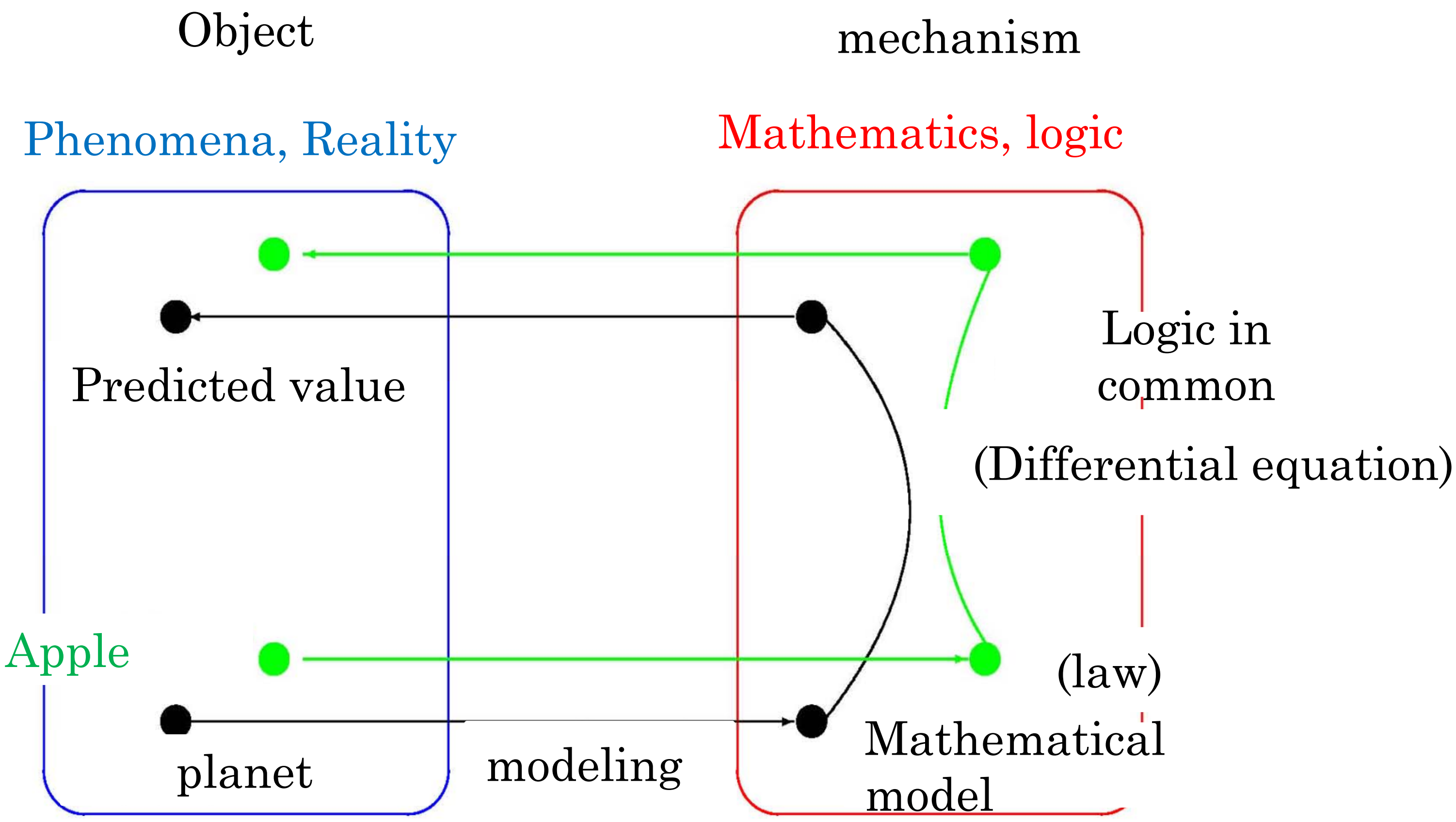
http://www.misojiro.t.u-tokyo.ac.jp/~murota

1. Mathematics in engineering

2. Mathematics in Optimization

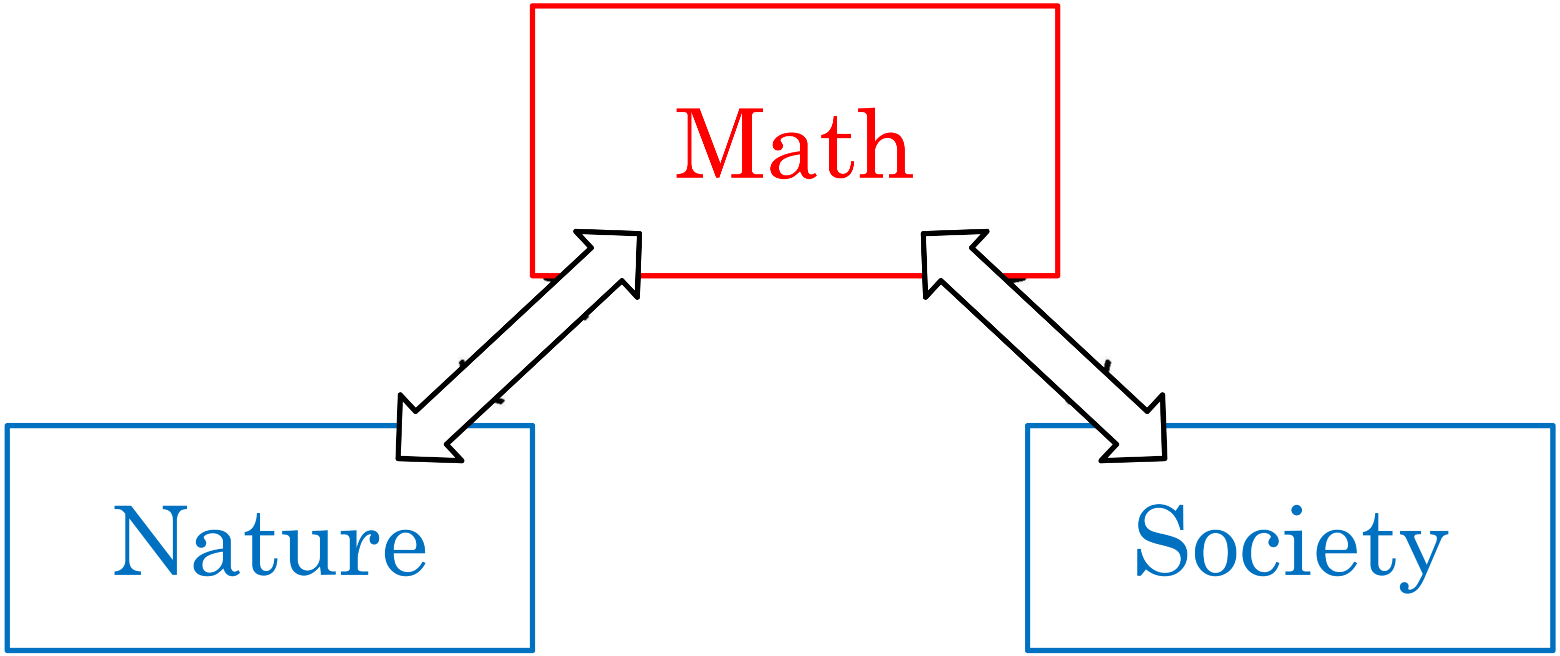
3. Models and Data

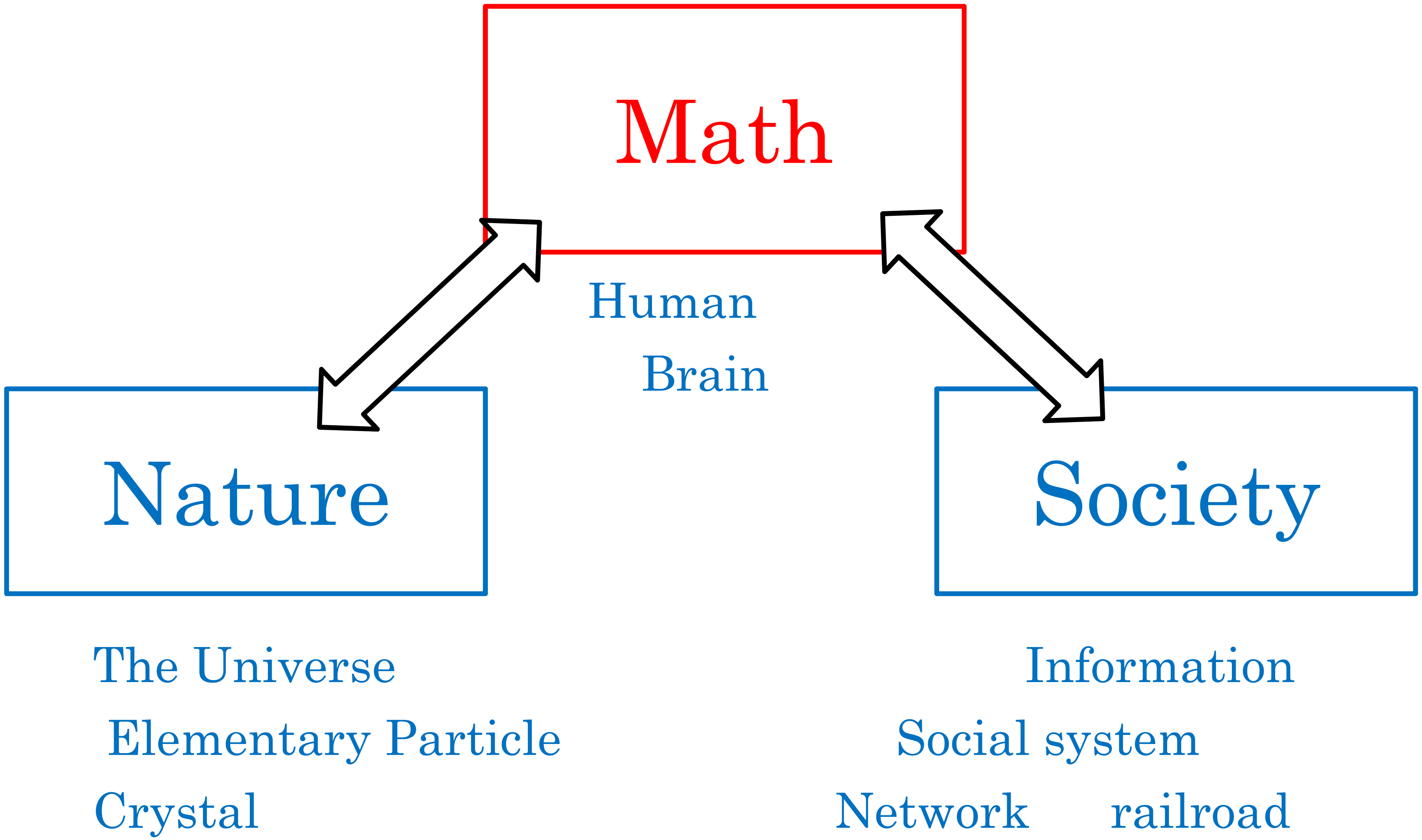
Phenomena and Mathematics

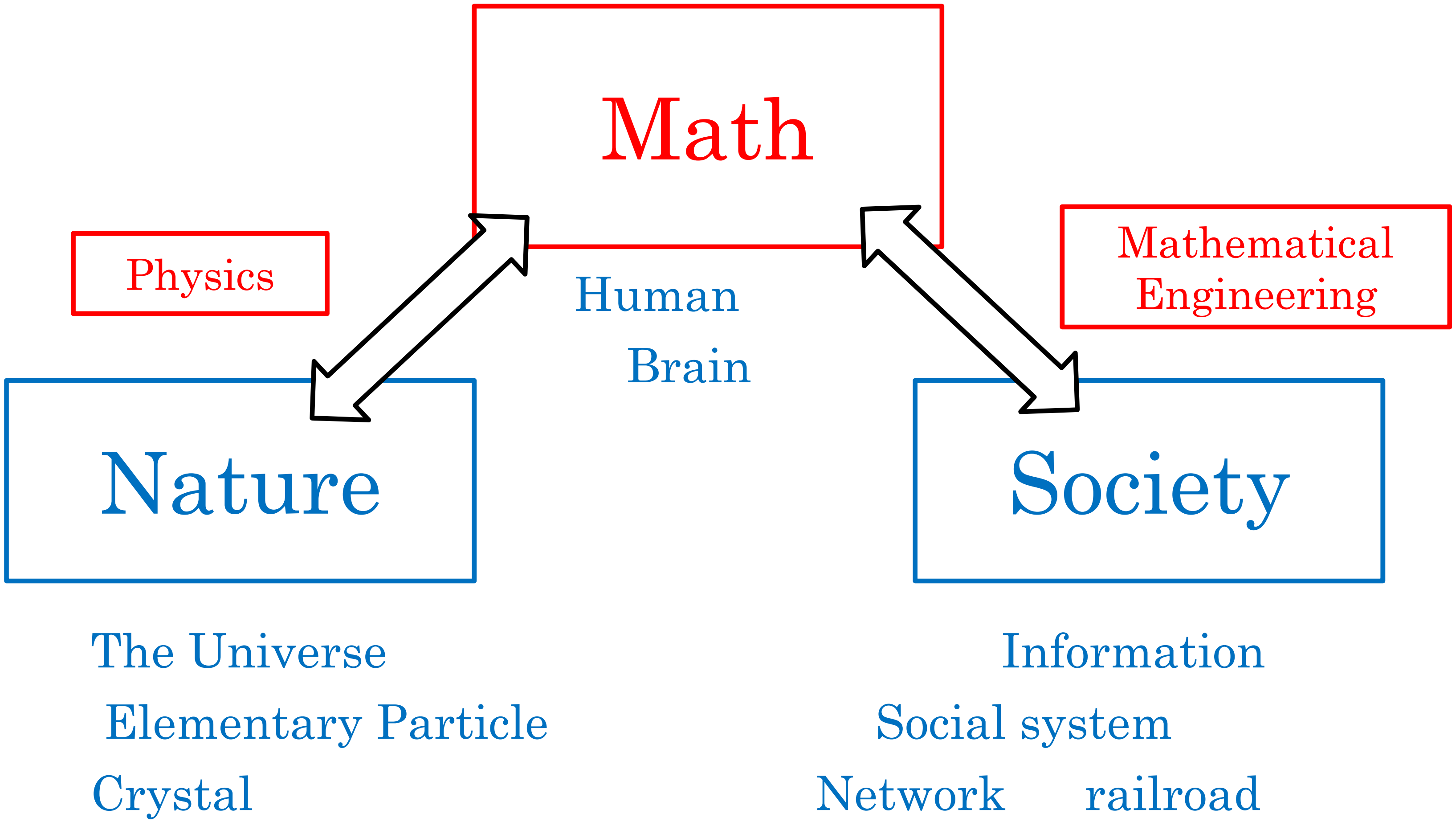


1. Mathematics in engineering

Driving force for mathematics




Math
Human
Brain
Nature
Society
The Universe
Elementary Particle
Crystal
Information
Social system
Network railroad


Math
Physics
Mathematical Engineering
Human
Brain
Nature
Society
The Universe
Elementary Particle
Crystal
Information
Social system
Network railroad

- Mathematics for Engineering appears around 1945

Around 1800	celestial mechanics	Gauss
1900	Axiomatization of mathematics	Hilbert
Around 1945	Mathematics for engineering	
2009	math, technology, and society today	

Computation

- What is computation? Gödel, Turing
- Principles for Computers Neumann

⇒Foundations of Mathematics, Computer Sciences, Numerical Analysis

Computation

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⇒Foundations of Mathematics, Computer Sciences, Numerical Analysis

Information

- Quantitative treatment of Information Shannon
- Cybernetics Wiener

⇒Information Theory, Brain Science, Data Science

Computation

- What is computation? Gödel, Turing
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- Cybernetics Wiener

⇒Information Theory, Brain Science, Data Science

Planning/ Decision

- Linear Programming Dantzig
- Game Theory von Neumann, Morgenstern

⇒Optimization, Control Theory, Decision Making (OR)

Mathematics for Engineering: the Dawning

Around 1800	Celestial mechanics	Gauss
1900	Axiomatization of mathematics	Hilbert

Around 1930	the theory of Computation	Gödel, Turing
1944	Game Theory	von Neumann, Morgenstern
1945	Programmed computers	von Neumann
1947	Linear Programming	Dantzig
1948	Theory for Communication and Information	Shannon
1948	Cybernetics	Wiener

2. Mathematics for Optimization

Planning, Decision

Maximization / minimization problem

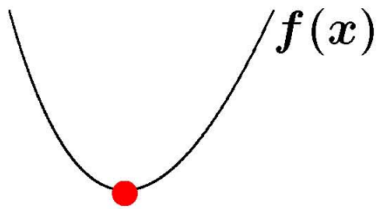
minimization $f(x)$

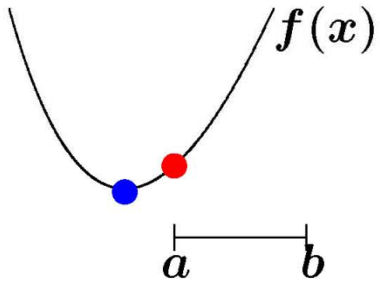
minimization $f(x)$ $\boxed{\Rightarrow \quad f'(x) = 0}$

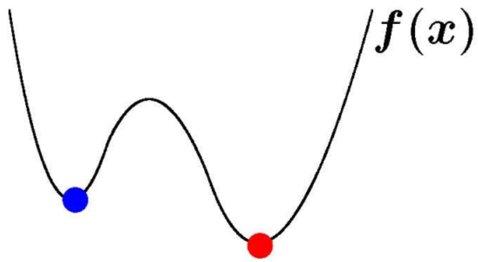
Constraint condition $a \leq x \leq b$

Minimization $f(x_1, x_2, \ldots)$

Constraint $(x_1, x_2, \ldots) \in \mathbf{S}$



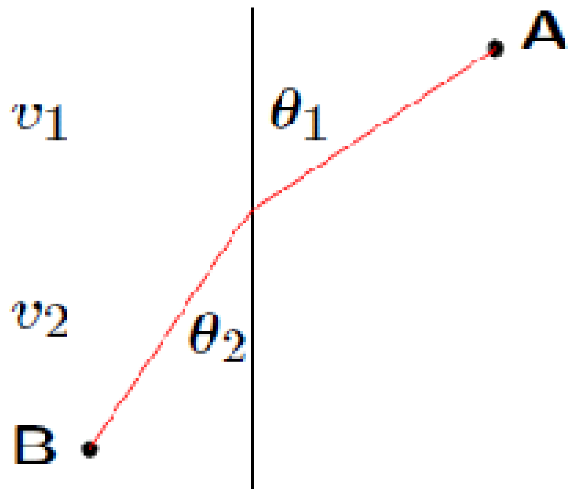




maximum minimum

Nature Chooses the Best— Variation Principle

Snell's Law: $$\frac{\sin\theta_1}{\sin\theta_2} = \frac{v_1}{v_2}$$



Fermat's Principle: the path taken between two points by a ray of light is the path that can be traversed in the least time

- least-action principle: motion path = minimization of action
- energy principle: equilibrium state = minimization of energy

Man Chooses the Best— Optimum Design

0 Series (1964) ‡ 300 Series (1992) ‡ 700 Series (1999) ‡

By courtesy of Akio Sagawa, Railway Technical Research Institute

Mechanical design= optimization that enables ability ↑ cost

Mathematical expression (modeling)
⇒ calculation ⇒ experiment ⇒ manufacturing

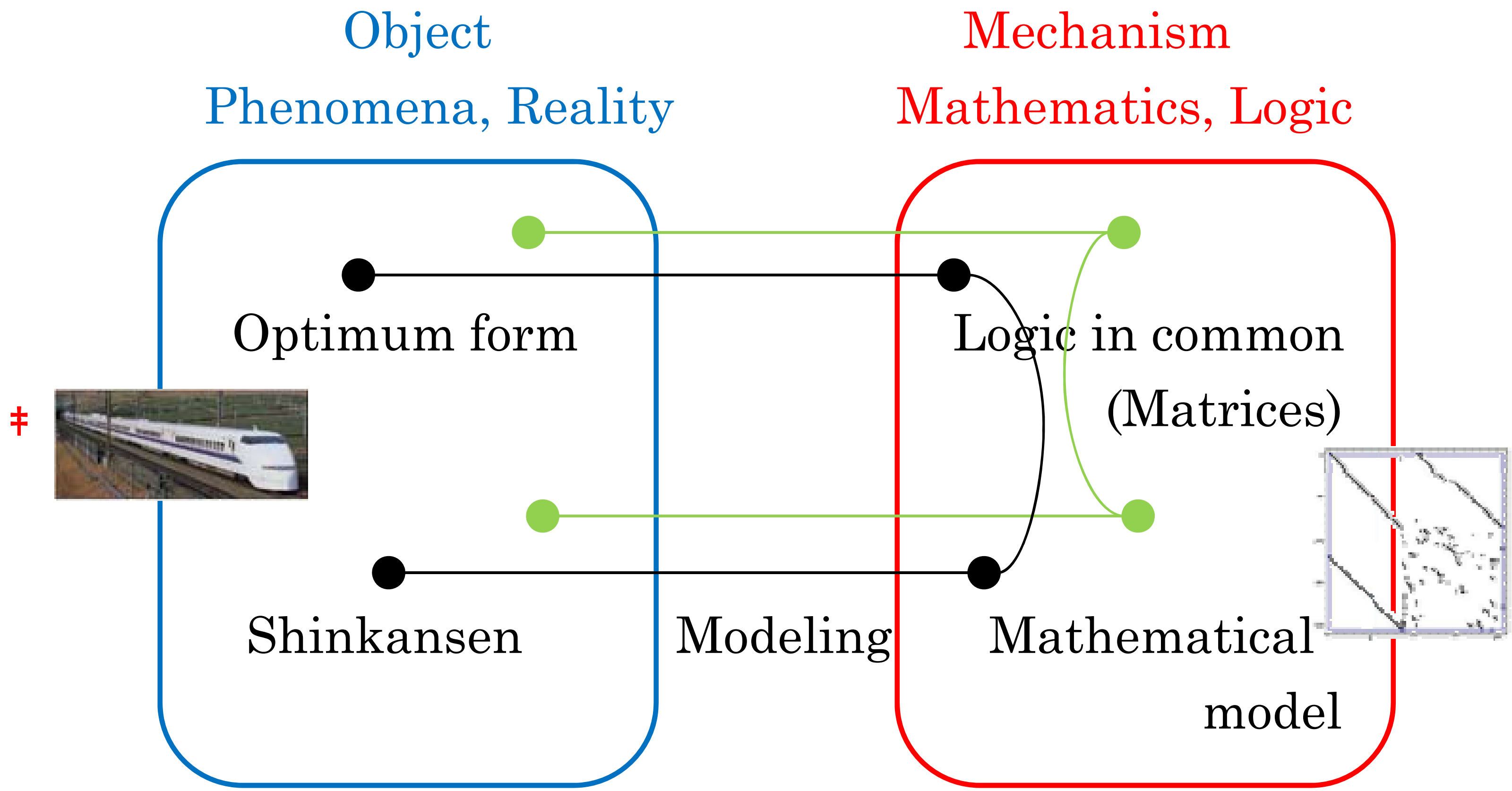
- **Man choose the optimum …**

  Optimum Decision Making (Re)

- How to get to Hongo ‡
  from Komaba?
  (13.7 km)

Goo map

Phenomena and Mathematics



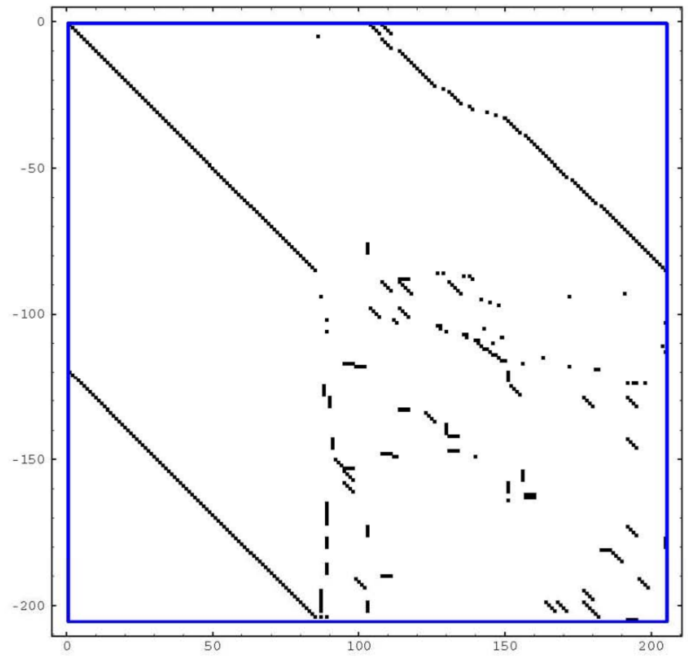
Matrices: common form

$$n = ?$$

Math textbook

$$\begin{bmatrix} 2 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 3 & 2 \end{bmatrix}$$

Actual problem in Engineering



White elements are all zero →

Large-scale, sparse matrix

- The World of Optimization

Operations Research

Mathematical programming

Modeling + Theory + Algorithm

Beautiful math　　　　Useful math

Birth of Mathematical Programming

1947: Linear Programming

‡ 

‡ 

‡ 

G. Dantzig (1914~2005)

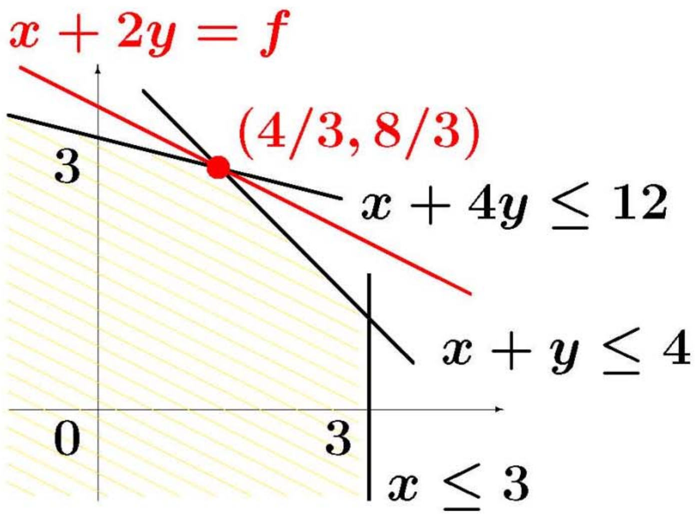
J. von Neumann (1903~1957)

L. V. Kantorovich (1912~1986)

http://www2.informs.org/Press/GeorgeDantzig.jpg
http://phil.elte.hu/redei/Utrecht/UtrechtNeumann.html
http://www.matematycy.interklasa.pl/images/matematycy/kantorovich.jpg

A Linear Programming problem in two variables

$$\begin{array}{lll} \textsf{Maximize} & x + 2y & =: \; f \\ \textsf{subject to} & x + 4y & \leq \; 12 \\ & x + y & \leq \; 4 \\ & x & \leq \; 3 \end{array}$$



Solution: $(x, y) = \left(\frac{4}{3}, \frac{8}{3}\right)$

Value: $f = \frac{4}{3} + 2 \cdot \frac{8}{3} = \frac{20}{3}$

General form of Linear Programming Problems

$$\begin{array}{ll} \textbf{Maximize} & c^\top x \\ \textbf{subject to} & Ax \le b \end{array}$$

$$\begin{array}{llll} \textbf{Max.} & x_1 + 2\ x_2 & & \\ \textbf{s.t.} & x_1 + 4\ x_2 & \le & 12 \\ & x_1 + \ \ x_2 & \le & 4 \\ & x_1 & \le & 3 \end{array}$$

Modeling: Linear Approximation is efficient for practical use

Theory: Duality Theorem

Algorithm: Simplex Method

Applications to Industrial society ⟺ New aspects of Linear algebra

An example of "How Mathematics was Created"

Duality in Linear Programming Problems

-The meaning of transposed matrices

Primary Problem	Dual Problem
Max. $c^\top x$	Min. $b^\top y$
s.t. $Ax \leq b$	s.t. $A^\top y = c$
	$y \geq 0$

$$
\begin{array}{llll}
\text{Max.} & x_1 + 2x_2 & & \\
\text{s.t.} & x_1 + 4x_2 & \leq & 12 \\
 & x_1 + x_2 & \leq & 4 \\
 & x_1 & \leq & 3
\end{array}
\qquad
\begin{array}{llll}
\text{Min.} & 12y_1 + 4y_2 + 3y_3 & & \\
\text{s.t.} & y_1 + y_2 + y_3 & = & 1 \\
 & 4y_1 + y_2 & = & 2 \\
 & y_1,\ y_2,\ y_3 & \geq & 0
\end{array}
$$

Duality Theorem: Max. of Primary Problem = Min. of Dual Problem

Let us Solve a Dual problem

$(u, v, w) = (y_1, y_2, y_3)$

$$\begin{array}{lrcll}
\text{Minimize} & 12\,u + 4\,v + 3\,w & =: & g & \cdots (1) \\
\text{subject to} & u + v + w & = & 1 & \cdots (2) \\
 & 4\,u + v & = & 2 & \cdots (3) \\
 & u,\ v,\ w & \geq & 0 & \cdots (4)
\end{array}$$

(3) $\rightarrow$ $v = 2 - 4u \qquad \geq 0$

(2) $\rightarrow$ $w = 1 - u - v = -1 + 3u \qquad \geq 0$

(1) $\rightarrow$ $g = 12u + 4(2 - 4u) + 3(-1 + 3u) = \boxed{5 + 5u}$

(4) $\rightarrow$ $\boxed{1/3 \leq u \leq 1/2}$

Solution: $(u, v, w) = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$ Optimal Value: $g = 5 + 5 \cdot \frac{1}{3} = \frac{20}{3}$

- **Let us Solve a Dual problem**

$(u, v, w) = (y_1, y_2, y_3)$

$$
\begin{array}{llcll}
\textsf{Minimize} & 12\,u + 4\,v + 3\,w & =: & g & \cdots(1) \\
\textsf{subject to} & u + v + w & = & 1 & \cdots(2) \\
 & 4\,u + v & = & 2 & \cdots(3) \\
 & u,\ v,\ w & \geq & 0 & \cdots(4)
\end{array}
$$

$$\textbf{(3)} \rightarrow v = 2 - 4u \qquad \geq 0$$

$$\textbf{(2)} \rightarrow w = 1 - u - v = -1 + 3u \qquad \geq 0$$

$$\textbf{(1)} \rightarrow g = 12u + 4(2-4u) + 3(-1+3u) = \boxed{5 + 5u}$$

$$\textbf{(4)} \rightarrow \boxed{1/3 \leq u \leq 1/2}$$

Solution: $(u, v, w) = \left(\frac{1}{3}, \frac{2}{3}, 0\right)$ Optimal Value: $g = 5 + 5 \cdot \frac{1}{3} = \frac{20}{3}$

Max. of Primary Problem = 20/3 = Min. of Dual Problem

Progress in Optimization theory (continuous variables)

1947	$\boxed{\text{Linear Programming}}$	Dantzig
1960	Nonlinear Programming, Newton method	Powell, Fletcher
1970	Convex analyses/ Duality Theory	Rockafellar
1979	Ellipsoid method	Khachiyan
1984	Interior Point Method	Karmarkar
1995	Semidefinite Programming	Alizadeh, Nesterov, Nemirovski

Theory: Linear/ Convex / Nonlinear

Environment: Increase in Computational Power

(reappear)

- **Man choose the optimum …**

Optimum Decision Making

- How to get to Hongo
  from Komaba?
  (13.7 km)

Goo map

Mathematics for Engineering (Re)

Computation

- What is computation?
- Principles for Computers
- Computational Efficiency

Information

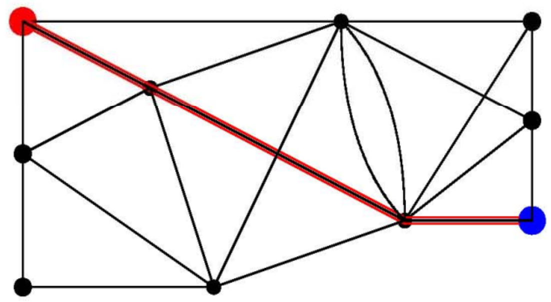
- Quantitative treatment of Information
- Cybernetics

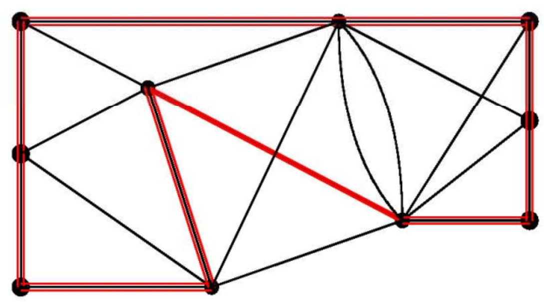
Planning/ Decision

- Linear Programming
- Game Theory

Discrete Optimization

Continuous/Discrete





Shortest Path Problem

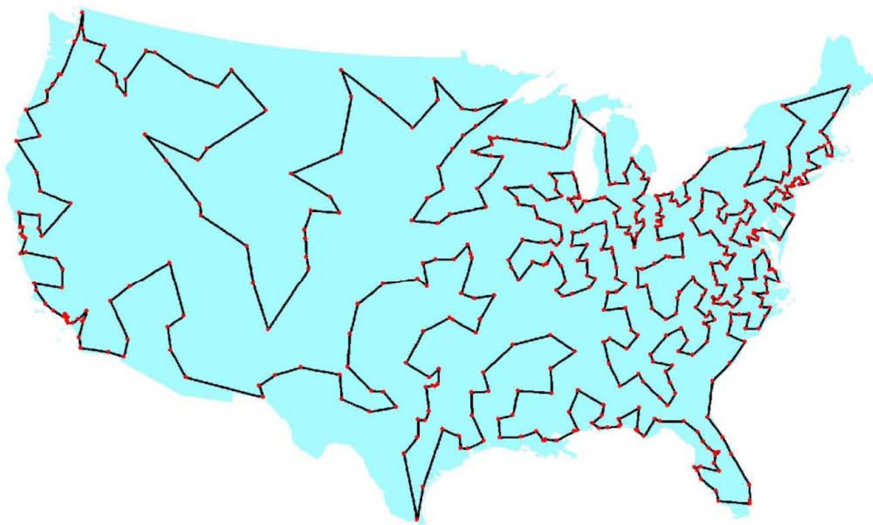
Traveling Salesman Problem

easy	(empirical fact)	very difficult
P	(algorithmic theory)	NP hard

Shortest Path Problem — Traveling Salesman Problem

easy to solve — difficult to solve

⇒demonstration (by Nobuyuki Tsuchimura)



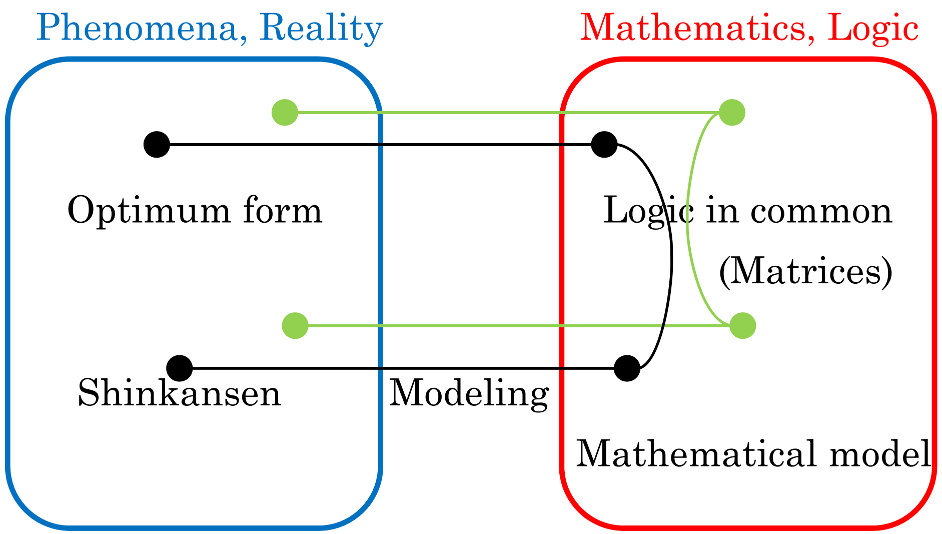
532 cities, M. Padberg–G. Rinaldi (1987)

http://www.tsp.gatech.edu/history/pictorial/att532.html

Progress in Discrete Optimization

1935	Matroid	Whitney
1947	Linear programming (Simplex method)	Dantzig
1956	Flow network	Ford, Fulkerson
1960	Integer Programming (Cutting plane method)	Gomory
1970	NP Complete	Cook, Levin
	Graph algorithm	Tarjan
	Submodular Function	Edmonds
1980	Ellipsoid method	Khachiyan
1995	Approximation Algorithm	Goemans, Williamson
2000	Discrete Convex Analysis	Murota

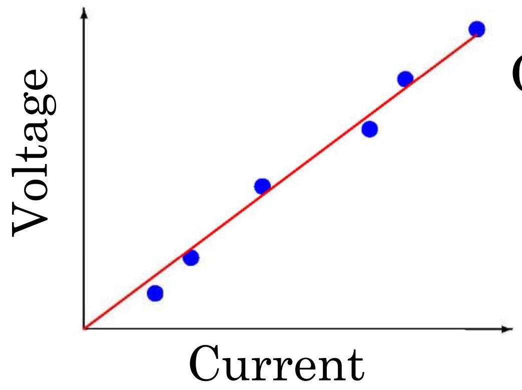
Models and Data



Models and Data

(Phenomenon⇔Math)

$$V = RI$$

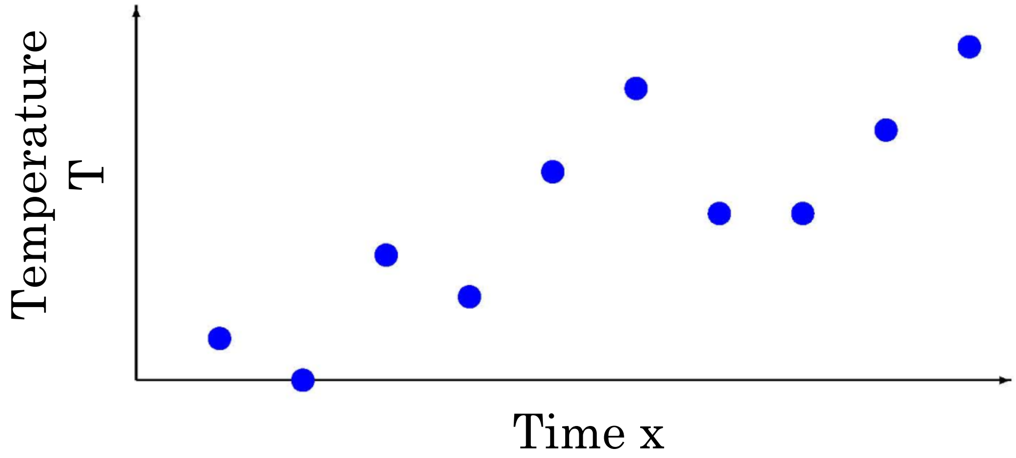


Ohm's law

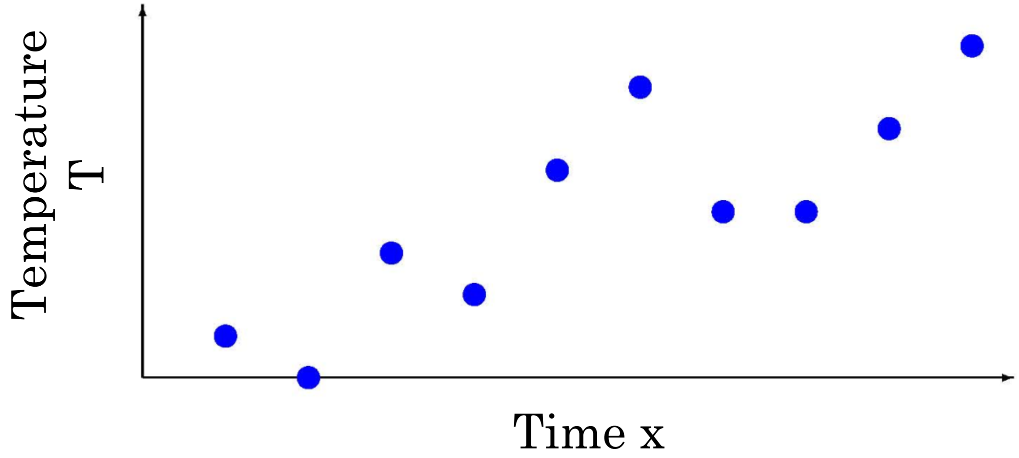
Collect experimental data and discover the true model

Least-square method (Gauss) : $$f(R) = \sum_i (V_i - RI_i)^2 \rightarrow \min$$

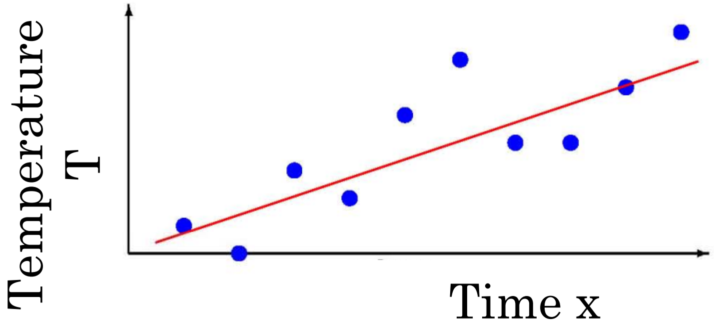
Model Fitting



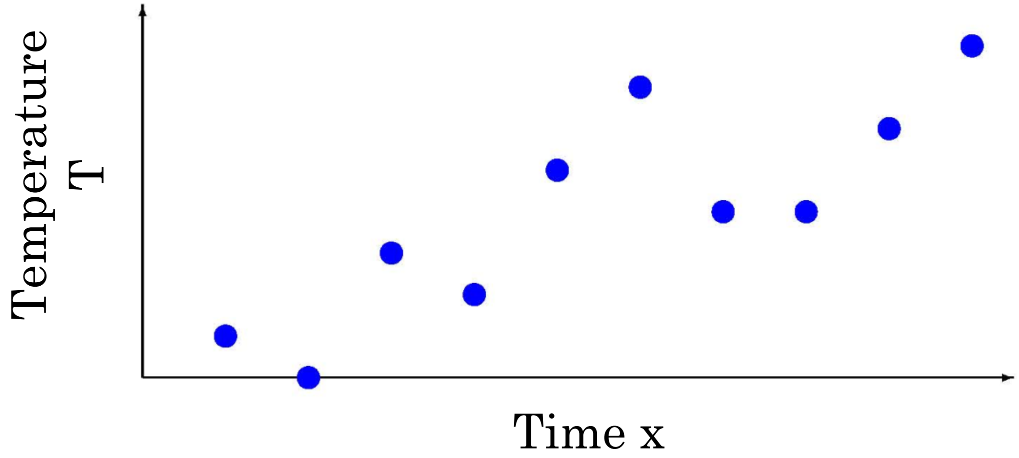
Model Fitting



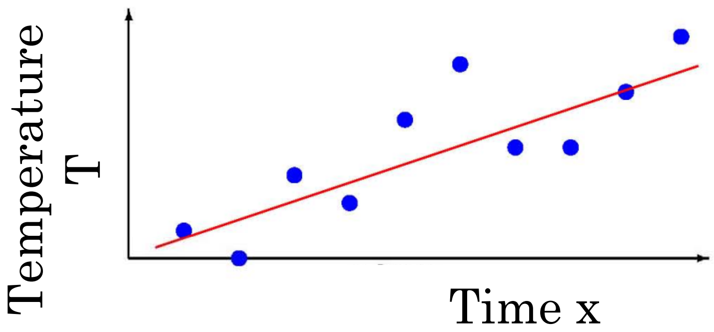
Linear function $T = ax + b$



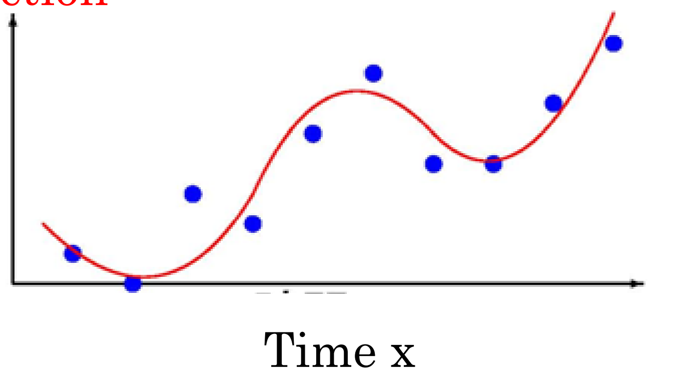
Model Fitting



Linear function $T = ax + b$

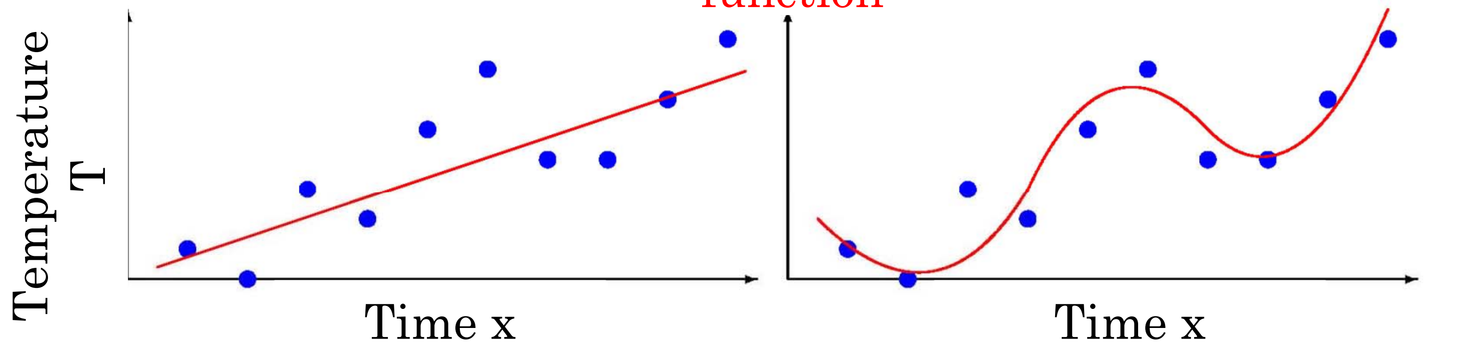


Quartic function $T = ax^4 + bx^3 + \cdots$



What is a good model?

Linear function $T = ax + b$ Quartic function $T = ax^4 + bx^3 + \cdots$



Complex Models are:

- Consistent with the data (explaining the past)
- Sensitive to errors, instable (predicting the future)

Consistency with the data Simplicity of the model

Information Criterion

Akaike Information Criterion

(Consistency with the data) (Simplicity of the model)

AIC = -2 (log likelihood) + 2(the number of parameters)

- 1974 Hirotsugu Akaike (1927-2009)

$\Rightarrow$ Information Geometry
(Shun-ichi Amari)

The significance of mathematical model

Models are

- used to relate reality to mathematics
- not uniquely determined

Natural Sciences (Pure Science):

True Model　　the search for truth

Technological Sciences (Engineering):

Appropriate model　　means to achieve the purpose

$\Rightarrow$ modeling science

- Qualitative change of method for obtaining data

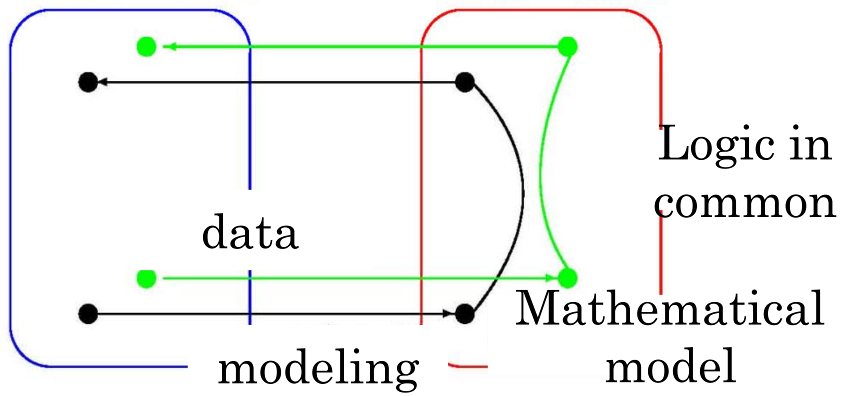
Past	Now (Web era)
make up by experiment / observation agricultural experiment	gather from around the world search engine (Yahoo!, Google)
Experimental planning (data gaining method) Analysis of variance (data analyzing method)	accumulative variance (information explosion) data mining

mathematical foundation: linear algebra (eigenvalue),
the concept of probability…

- Summary:

  Mathematics <Mathematical Engineering> society

Math for technology



Computation

Information

Planning, Decision

Mathematics for Optimization

Modeling + Theory + Algorithm

The meaning of mathematical models,

Qualitative change of data

Next lecture …